Analysis of All the End Game Possibilities in Catch-up When There Are 3 or 4 Pieces Remaining

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Abstract

Catch-up is a two-player game where the turn alternates only when a player ties or takes the lead. Unlike games where a lead gives you an advantage over your opponent, in Catch-up a lead cannot be maintained past a single turn. This is interesting because the score remains close throughout, making it difficult to determine who is in a winning position. In this paper, the end game possibilities when there are three or four pieces remaining to choose from are analyzed. This is done using a comprehensive case-by-case study. This way, we gain a deeper knowledge of the game's strategy.

1 Introduction

1.1 Game Theory

Game theory studies the interaction of the decisions made by the players, the strategies they use, and the outcome as the consequence of said decision.[3] Our focus is on games that do not possess any element of chance and no hidden information to the players. There exist some such games where the outcomes have been explored entirely. These include Tic Tac Toe, Connect 4, Mancala Awari, and others. This means that all the results of the game and how to get there are known, so if a player know the ideal moves, it will always lead to a tie or a win for that player. Such games are called solved. A possible reason why these kinds of games are solved and some are not is because of the complexity of the game. For example, Tic Tac Toe has a small set of possible moves, and Connect 4's structure is simple enough to be studied. On the other hand, there are games where the solution has not been found yet. The Fundamental Theorem of Game Theory states that if a game has no element of chance and perfect information, then there exist either a winning or drawing strategy for one of the player. This means that such games have a solution, even if it has not yet been found. Some examples of such unsolved games are chess and Go. But these games have been studied for many years and strategies have come up from trying to find the solution for these games. As a result, there are openings and a deeper understanding of winning and losing positions, and in end games, there is intuition for the outcome based on the game state.

A Game tree is a mathematical object used in game theory to study decision making and consequences.[1] The game tree contains vertices and edges, vertices represented game states, and edges which represent actions a player could take. The vertices are connected by the edges because depending on what the player decide to do in their turn, they could end up in a different vertices because they could travel from a different edge. A game tree is a graphical diagram that provide information about all the possible actions that players can take and their consequences.

Game theory is not limited to board games. The book [2] discussed the usage of game theory as a framework to help explore how different voting systems affect the decision-making of the voters and the outcome of the votes. For example, when the voting system is limited to one vote, this means voters believe their vote is important and want to choose the person they think would do the job best, in other words, they do not want to waste their vote. But if we twist the system to more than one vote allowed, now voters can choose multiple people including the best candidate and some candidates they feel can do a decent job as well, this leads to less risk in the voters' minds. There are systems where they can vote for the winner and some preferred candidates. This is a way to spread information to the voters and raise candidates' popularity in the future. On the other side, depending on the system, the people running for office would act according to how voters think. This is done because candidate wants to get the best chance at winning so they will tackle the weaknesses of a voting system to get the most votes. Therefore, the government or the people creating the voting system are responsible for choosing the one that is the most fair and balanced in terms of power and winning chances for all voters and candidates. This can tie back to game theory in board games by discussing the idea that depending on the game's design and its rules, players will apply strategies and make decisions based on the structure of the game. They act accordingly to increase their chances of winning. As a result, designers of games need to consider this when creating a game if they want all players to have the same chances to win the game.

1.2 Rules

Catch-up is a two player (P_1 and P_2) game where players take turns to take numbers from a set S of natural numbers. Our focus is on the set S_N, the consecutive positive integers from 1 to N, $S_N = \{1, 2, ..., N\}$. Players select numbers from S_N which are added to their score. Players need to decide two elements of the game before playing: who has the first move to start the game and what value of N they want to play, in other words, how large they want the set S_N to be. After this players start choosing numbers one at a time, a player's turn ends when their score match or exceeds the opponent's score. This way no player can maintain a higher score than the other. The game ends when players have chosen the entirely of S_N. The player with he highest score wins.

For example, lets assume P_1 has the first move and P_2 has the second move. The game start with 0 as the score for each player. P_1 can choose any number from S_N, then the turn switches to P_2. Then P_2 can choose any number from S_N, the turn remains in P_2's hand until P_2's score is the same or greater than P_1's score. Then the turn switches back to P_1 and P_1 can choose any number from S_N until P_1's score is the same or greater than

P_2's score. They repeat this procedure until there is no more numbers to choose from. The winner is the player with the highest score at the end, or if the score is tie, then there is a draw.

1.3 Example of a Play-through

At the start of a game, players decide on what N they want to play. Suppose $N = 4$. It is the start of the game so no player has done a move yet to get a point. The game start with a set of number in this case is numbers 1 through 4.

The starting game state is

$\{1,2,3,4\}$ and the score is 0 to 0.

P_1 chooses 2

The game state is

$\{1,3,4\}$ and the score is 2 to 0.

P_2 chooses 1

The game state is

$\{3,4\}$ and the score is 2 to 1.

Since the score for P_2 has not matched or exceed P_1, it is still P_2's turn.

P_2 chooses 3

The game state is

$\{4\}$ and the score is 2 to 4.

Now that P_2's score exceeded P_1's, the turn switches to P_1.

P_1 chooses 4

The game state is

$\{\}$ and the score is 6 to 4.

The game ends when all numbers have been chosen, and P_1 wins because they have a higher score at the end.

1.4 Background

The article [4] describes the basic rules of the game, its properties, and the reasons for the complexity of this simple game.

The article focuses on the set S_N of consecutive positive integers from 1 to N. As defined before, the game tree size of Catch-up is the number of unique play-throughs. It is shown in the article that the size of a game tree when starting with set S_N is exactly $N!$. This characteristic makes Catch-up challenging to explore because of its magnitude since N is increasing in factorial. The researchers from the article investigated up to $N = 20$ when by optimal play, one of the players force a win or draw. For example, they found that when $N = 5, 6, 13, 17$, the player with the first move has a strategy to force a win, and when $N = 9, 10, 14, 18$, the player with the second move has a strategy to force a win, and when $N = 3, 4, 7, 8, 11, 12, 15, 16, 19, 20$, the player with the first move has a strategy to force a draw; in other words, there are no strategies to force a win. They collected this data by brute force: all the outcomes and results were generated by a computer which allowed them to solve those games depending of the value of N, determining which player has a winning of drawing strategy. By inspecting the values of N, there is a pattern for when a player can force a win or a draw, and the authors conjecture that this pattern continues for all n. But because of the nature of the game tree size which is $N!$, it gets incredibly large rapidly, preventing researchers to explore the game for $N > 20$.

Heuristics are strategies that help players learn how the game works and how to play the game. They usually are general strategies to the game. The goal is to make it easier for players to apply them when judging what the best move is when playing the game. Some examples of heuristics from the article are choosing the highest number in the set, choosing the lowest number in the set, choosing the most amount of numbers in the set, or a combination of these. They were not able to find a dominant heuristic that is superior to others, so they emphasized that there is room for more innovation in this department by the players to come up with a set of heuristics that they believe can get them an advantage over their opponent.

Some properties that the article explained are total points that can be scored in a game based on N, maximum numbers of points in one turn, possible numbers of moves in a given game state when there exists a draw, and others. These properties are useful because they can help you build intuition and provide a better understanding of how the game works.

The nature of the game makes it difficult to predict who is going to win early on in the match. Prediction becomes easier during the last few moves of the game, when there are few numbers left to choose from. This feature of Catch-up is what makes it interesting yet tough to analyze. The main goal of research on Catch-up is to prove the existence of a winning strategy and techniques on how to explore the game for any value of N. But for this, a more creative approach needs to be fabricated in order to explore the game more in-depth.

1.5 Motivation

It would be helpful to know the outcomes of the games in the end game, but we cannot memorize all of them because they are so many. So understanding how the game works in the end game is important to know how to win the game. Optimal play in a game where the size of the game tree is $N!$ is incredibly difficult and not realistic because the number of possible moves is enormous. Similar to chess where there are so many legal moves in any game state, the judgment of the optimal play is too difficult. Therefore, optimal play throughout the whole game is achievable mostly by computer because they can calculate all the possible outcomes of any action, analyze them very quickly, and determine the best move. For this reason, players need to create heuristic strategies to gain an advantage over their opponents. But this is not easy because Catch-up is a game where you cannot maintain a lead, which means it is challenging to understand who is winning in a given game state. In this article, we analyze who has a winning strategy during the end game when there are three or four numbers remaining.

Analyzing all the end game outcomes when there are three or four numbers left in the set to choose from can help players understand how the game works. By knowing this, there might be ways to get into the end game with an advantage over your opponent if you know what is the winning condition that you are playing for. This can be done by observing patterns that signify wins, draws, or losses during the said analysis.

2 Results

Recall that $S_N = \{1, 2, ..., N\}$. In the article "Catch-Up: A Game in Which the Lead Alternates." by Steven J. Brams, et al. uses that same set. It is important to note that the results of our research can be applied for any set S of positive consecutive or non-consecutive integers.

- Let P_1 represent player one and P_2 represent player two.
- Let S represent the initial set of numbers.

- Let S_N be the set of positive consecutive integers from 1 to N. That is, $S_N = \{1, 2, ..., N\}$.
- Let D be the difference between the player's scores. That is, $D = P_1$'s score minus P_2's score. This means that when D switches sign or is 0, it signifies to end a turn. Also, when D is positive P_1 is winning and when D is negative P_2 is winning.
- Let $B = \{b_1, b_2, ..., b_n\}$ be the set of remaining numbers a player can choose from during any given turn, where n is the number of remaining numbers and $b_1, b_2, ..., b_n$ are in increasing order.
- Let $(\{B\}, D)$ represent a game state.

Theorem 1 *If $D = 0$ and $n = 3$ or $n = 4$, then there exist no winning strategy for P_2 because P_1 always have a strategy to either force a win or a draw.*

Theorem 2 *Suppose $D \neq 0$ and $n = 3$. P_2 has a winning strategy **if and only if** one of the following conditions are met:*

1. $|D| > b_2$, $b_2 + b_3 < b_1 + |D|$, *and* $b_1 + b_2 + b_3 < |D|$,
2. $|D| > b_2$, $b_2 + b_3 < b_1 + |D|$, *and* $b_1 + b_2 \geq |D|$,
3. $|D| \leq b_2$, $b_1 < |D|$ *and* $b_1 + b_3 < b_2 + |D|$,
4. $|D| \leq b_2$, $b_1 > |D|$, $b_1 + b_3 < b_2 + |D|$ *and* $b_3 \leq b_1 + b_2 + |D|$,
5. $|D| \leq b_2$, $b_1 = |D|$, $b_1 + b_2 + |D| > b_3$ *and* $|D| - b_3 > b_1$.

After providing the comprehensive endgame analysis, we will highlight which aspect culminate in the proofs of these theorems.

2.1 Method

The objective of this research is to find all the cases that exist when $n = 3$ and $n = 4$ and to analyze the results. In order to do this, it is necessary to understand how the end game of Catch-up works. When playing the game informally, most of the time the game does not end in a draw. There usually is a winner and a loser. This suggest that there usually exists a winning strategy for some player.

To study this, I played the game numerous times to gather information on when someone won, lost, or drew. By doing this, I were able to identify patterns and construct general cases. Afterward, I filled in all the gaps from the discovered scenarios to cover all the cases that are achievable in the game, I did this using inequalities to cover all the possibilities and be able to generalized the outcomes. Moreover, with all the cases built, I generated random game states that satisfy the case's characteristics. This way, I can investigate by playing the case over and over again with all possible moves and find out what the end result most likely is. This is possible because the game tree is relatively small for us to play around when B is narrowed down to $n = 3$ or $n = 4$. In the end, I stated what action produced what outcome and identified the result of the game that is favorable, which is the wins, or at worst the draws. Note that in the results, P_1 and P_2 are not necessarily the player who went first or second at the start of the game, but the player with the first and second move from the moment indicated.

I created a list of all the possible cases when there are three or four pieces remaining, in terms of relationships between the b_1, b_2, b_3 (and b_4) and D, and the best result possible by playing the specific move. This helps us predict the outcome of the games given any game state.

2.2 Analysis of the End Game:

2.2.1 When $D = 0$ and $n = 3$

Let P_1 be the player to move next when B is narrowed down to $\{b_1, b_2, b_3\}$ and let P_2 be the other player. This means P_1 is not necessarily the player who went first at the start of the game. Let $D = P_2$'s score minus P_1's score with the new definition of P_1 and P_2 where if D is positive, the lead belongs to P_2 and if D is negative, the lead belongs to P_1.

I. If $b_1 + b_2 > b_3$, then choose b_1. P_1 Wins.

   By choosing b_1 first, this forces P_2 to choose at most 1 number from B ending their turn. This way, P_1 can have either b_1 and b_2 which P_1 wins because $b_1 + b_2 > b_3$ or b_1 and b_3 where P_1 also wins because $b_3 > b_2$.

   Example: $(\{5, 7, 9\}, 0)$

II. If $b_1 + b_2 < b_3$, then choose b_3. P_1 Wins.

   By choosing b_3, no matter if P_2 gets the all the remaining numbers b_1 and b_2, since $b_1 + b_2 < b_3$ P_1 wins with b_3 by itself.

   Example: $(\{1, 2, 5\}, 0)$

III. If $b_1 + b_2 = b_3$, then choose b_1. P_1 draws but could win if P_2 makes a mistake.

   P_1 does not choose b_3 for a forced draw because we want to give a chance for P_2 to make mistakes, so we have the possibility to either win or at the worst draw. After choosing b_1, if P_2 chooses b_3, it is a draw. But, if P_2 choose b_2, P_1 wins by getting b_3 and b_1 where $b_3 > b_2$ by itself.

   Example: $(\{1, 2, 3\}, 0)$

2.2.2 When $D = 0$ and $n = 4$

Let P_1 be the player to move next when B is narrowed down to $\{b_1, b_2, b_3, b_4\}$ and let P_2 be the other player. This means P_1 is not necessarily the player who went first at the start of the game. Let $D = P_2$'s score minus P_1's score with the new definitions of P_1 and P_2 where if D is positive, the lead belongs to P_2 and if D is negative, the lead belongs to P_1.

I. If $b_2 + b_3 > b_1 + b_4$, then choose b_2. P_1 wins.

   After P_1 has chosen b_2, P_2 cannot choose b_1 and b_4 to get the most amount of points in one turn because P_2 loses by the hypothesis that $b_2 + b_3 > b_1 + b_4$. P_2 can choose b_3 first, but P_1 would get b_4 where $b_4 > b_3$ and $b_2 > b_1$ and P_1 wins with b_4 and b_2. P_2 can choose b_4, then P_1 end up with b_3 right after and win by the same hypothesis.

   Example: $(\{1, 4, 5, 6\}, 0)$

II. If $b_2 + b_3 < b_1 + b_4$, then consider the following cases:

   A. If $b_1 + b_2 + b_3 > b_4$, then choose b_1. P_1 wins.

      Here P_1 cannot choose b_2 because P_2 would get b_1 and b_4 and P_1 loses because of the hypothesis $b_2 + b_3 < b_1 + b_4$. So P_1 would choose b_1 to limit P_2 by making them able to choose at most 1 number. If P_2 chooses b_2 or b_3, P_1 would get b_4 and win by the same hypothesis. If P_2 chooses b_4, P_1 gets b_2 and b_3 and wins because $b_1 + b_2 + b_3 > b_4$.

      Example: $(\{2, 3, 4, 7\}, 0)$

   B. If $b_1 + b_2 + b_3 < b_4$, then choose b_4. P_1 wins.

      By choosing b_4 no matter if P_2 gets the all the remaining numbers b_1, b_2 and b_3, since $b_1 + b_2 + b_3 < b_4$ P_1 wins with b_4 by itself.

      Example: $(\{1, 3, 4, 10\}, 0)$

C. $b_1 + b_2 + b_3 = b_4$, then choose b_1. P_1 draws but can win if P_2 makes a mistake.

P_1 cannot start with b_2 or b_3 because P_2 would get b_1 and b_4, since $b_1 + b_2 + b_3 = b_4$ P_1 would lose. So, P_1 has the option to choose b_4 to force a draw. But P_1 can also choose b_1 and if P_2 chooses anything but b_4, P_1 would win because P_1 would be the one getting b_1 and b_4 and win.

Example: $(\{3, 4, 5, 12\}, 0)$

III. If $b_2 + b_3 = b_1 + b_4$, then choose b_2. P_1 draws but can win if P_2 makes a mistake.

This is a force draw if P_2 chooses b_1 and b_4. If P_2 chooses b_3 or b_4, P_1 wins the game by choosing b_1 and whatever value is left in B. This way, P_1 ends up having b_1, b_2 and b_3 or b_4.

Example: $(\{3, 5, 6, 8\}, 0)$

2.2.3 When $D \neq 0$ and $n = 3$

Let P_1 be the player to move next when B is narrowed down to $\{b_1, b_2, b_3\}$ and let P_2 be the other player. This means P_1 is not necessarily the player who went first at the start of the game. Let $D = P_2$'s score minus P_1's score with the new definitions of P_1 and P_2 where if D is positive, the lead belongs to P_2 and if D is negative, the lead belongs to P_1.

I. If $D > b_2$, then consider the following cases:

A. If $b_2 + b_3 > b_1 + D$, then choose b_2 and b_3. P_1 wins.

Since $D > b_2$, after choosing b_2 it is still P_1's turn, so P_1 can choose b_3 next. P_1 wins after obtaining b_2 and b_3 because $b_2 + b_3 > b_1 + D$.

Example: $(\{3, 4, 6\}, 5)$

B. If $b_2 + b_3 < b_1 + D$, then consider the following cases:

1. If $b_1 + b_2 + b_3 < D$, then choose any number. P_1 loses.

   No matter what P_1 chooses, even if P_1 gets all three numbers remaining, P_1 cannot have a higher score than what P_2 already has with D by itself.

   Example: $(\{1,2,3\},8)$

2. If $b_1 + b_2 + b_3 = D$, then choose b_1, b_2 and b_3. P_1 draws.

   P_1 cannot surpass P_2's score, since $b_1 + b_2 + b_3 = D$, the only scenario that can happen is a draw the moment P_1 chooses all the three remaining values in B.

   Example: $(\{2,3,4\},9)$

3. If $b_1 + b_2 + b_3 > D$, then consider the following cases:

   a. If $b_1 + b_2 \geq D$, then choose any number. P_1 loses.

      P_1 cannot win because since $b_2 + b_3 < b_1 + D$, P_2 only needs b_1 to win, which means b_2 or b_3 works for them as well. P_1 cannot choose all the remaining numbers all at once. Therefore, P_1 loses.

      Example: $(\{3,4,5\},7)$

   b. If $b_1 + b_2 < D$, then choose b_1 and b_2. P_1 wins.

      P_1 wins because $b_1 + b_2 < D$ which means after choosing b_1 and b_2, it is still P_1's turn. P_1 needs to get all the remaining values in B to win since $b_1 + b_2 + b_3 > D$, which means P_1 need to choose b_3 only as their last value. Otherwise, P_1's turn could end before getting everything.

      Example: $(\{2,3,4\},7)$

C. If $b_2 + b_3 = b_1 + D$, then choose b_2. P_1 draws.

P_1 cannot win because since $b_2 + b_3 = b_1 + D$, P_2 only needs b_1 to win, which means b_2 or b_3 works for them as well. P_1 cannot prevent P_2 from choosing any number in B. Therefore, P_1' best choice is to draw by choosing b_2 and b_3.

Example: $(\{5,6,7\},8)$

II. If $D \leq b_2$, then consider the following cases:

I. Suppose $b_1 < D$, then consider the following cases:

A. If $b_1 + b_3 > b_2 + D$, then choose b_1 and b_3. P_1 wins.

P_1 can choose b_1 and it is still their turn. So, P_1 can choose b_3 right after and win since $b_1 + b_3 > b_2 + D$.

Example: $(\{2,4,8\},3)$

B. If $b_1 + b_3 < b_2 + D$, then choose any. P_1 loses.

P_1 cannot win with these conditions because it is not possible to prevent P_2 from getting either b_2 or b_3. If P_1 chooses b_1 and b_3, then P_1 loses since the hypothesis is that $b_1 + b_3 < b_2 + D$. If P_1 chooses b_2, then it is P_2's turn and they can choose b_3 and win because they only need b_2 to win by the same hypothesis, so b_3 works even better. If P_1 chooses b_3, P_2 simply gets b_2 and wins by the same hypothesis.

Example: $(\{1,7,8\},3)$

C. If $b_1 + b_3 = b_2 + D$, then choose b_1 and b_3. P_1 draws.

Since $b_1 + b_3 = b_2 + D$, P_1 can draw the game right away because $b_1 < D$ which means it is still P_1's turn after choosing b_1. P_1 loses on the spot if P_1 chooses b_3 because P_2 gets everything else after that. P_1 could try and choose b_2, but if P_2 plays the correct moves, P_1 loses as well. So, the best option here is to draw.

Example: $(\{4,6,7\},5)$

II. Suppose $b_1 = D$, then consider the following:

A. If $b_3 > b_1 + b_2 + D$, then choose b_3. P_1 wins.

After P_1 chooses b_3, P_2 cannot surpass P_1's score even if they get all the remaining values in B, so P_1 wins with b_3 alone.

Example: $(\{2,3,9\},2)$

B. If $b_3 = b_1 + b_2 + D$, then choose b_3. P_1 draws.

P_1 has to prevent P_2 from getting b_3 but doing so will end the game in a draw. If P_1 starts with b_1 or b_2, P_2 gets b_3 and P_1 loses. So, P_1's best option is to go for the draw.

Example: $(\{3,4,10\},3)$

C. If $b_3 < b_1 + b_2 + D$, then consider the following cases:

1. If $D - b_3 \leq b_1$, then choose b_3. P_1 wins.

   By choosing b_3, P_1 is limiting P_2 to choose at most one value in their turn since $D - b_3 \leq b_1$, so P_2 can either choose b_1 or b_2. At the end, P_1 have either b_3 and b_1 or b_2. P_1 wins because consider having b_3 and b_1 which is the lowest combination of points P_1 would get. $b_1 = D$ so D is neutralized, and P_1 win because $b_3 > b_2$.

   Example: $(\{2,3,4\},2)$

2. If $D - b_3 > b_1$, then choose any number. P_1 loses.

   P_1 cannot win in these conditions because if P_1 chooses b_3, then P_2 would get b_1 and b_2 and P_1 loses since $b_3 < b_1 + b_2 + D$. If P_1 starts with b_1 or b_2, remember this ends their turn since $b_1 = D$, P_2 can choose b_3 and win the game since $b_3 > b_2$ and $b_1 = D$. So P_2 only needs b_3 to win.

   Example: $(\{3,4,9\},3)$

III. Suppose $b_1 > D$, then consider the following cases:

A. If $b_1 + b_2 > b_3 + D$, then choose b_1. P_1 wins.

By choosing b_1, P_1 is limiting P_2 to choose at most one value from B because D after choosing b_1 is always smaller than b_2. Since $b_1 + b_2 > b_3 + D$, P_1 only needs b_2 to win, which means b_3 also works. P_2 cannot choose both values at the same time so P_1 wins.

Example: $(\{6, 7, 8\}, 2)$

B. If $b_1 + b_2 = b_3 + D$, then choose b_1. P_1 draws.

By choosing b_1, P_1 is limiting P_2 to choose at most one value from B. P_2 has to choose b_3 resulting in a draw, if P_2 chooses b_2, P_1 wins. If P_1 starts the game with b_2 or b_3 instead, because P_1 is making the value of D higher, P_2 has the chance to get b_1 and b_2 or b_3 resulting in P_1's loss.

Example: $(\{2, 7, 8\}, 1)$

C. If $b_1 + b_2 < b_3 + D$, then consider the following cases:

1. If $b_3 > b_1 + b_2 + D$, then choose b_3. P_1 wins.

   No matter if P_2 gets all the remaining numbers in B, no sum of the remaining numbers is greater than or equal to b_3 by itself.

   Example: $(\{2, 3, 7\}, 2)$

2. If $b_3 = b_1 + b_2 + D$, then choose b_3. P_1 draws.

   Recall P_1 is limited to choosing at most one value because $b_1 > D$. P_1 cannot choose b_1 or b_2 because they would get b_3. Since P_2 also adds D to their score, P_1 would lose. The other option is to choose b_3, which would result in a draw since P_2 would get all the remaining values.

   Example: $(\{3, 5, 10\}, 2)$

3. If $b_3 < b_1 + b_2 + D$, then choose any number. P_1 loses.

   Again, P_1 is limited to choosing at most one value. If P_1 chooses b_3, then since $b_1 + b_2 < b_3 + D$, P_2 takes both b_1 and b_2 and P_1 would lose. If P_1 takes b_2, P_2 would get b_1 and b_3, resulting in a loss. For P_1, the only remaining option is to choose b_1. Then P_2 would get b_3, this is a loss for P_1 as well because $b_1 + b_2 < b_3 + D$.

   Example: $(\{2, 4, 6\}, 1)$

2.2.4 When $D \neq 0$ and $n = 4$

Let P_1 be the player to move next when B is narrowed down to $\{b_1, b_2, b_3, b_4\}$ and let P_2 be the other player. This means P_1 is not necessarily the player who went first at the start of the game. Let $D = P_1$'s score minus P_2's score with the new definition of P_1 and P_2 where if D is positive, the lead belongs to P_2 and if D is negative, the lead belongs to P_1.

I. If $D \leq b_1$

In this scenario, P_1 is limited to choosing at most only one value. Consider the following cases:

A. If $b_2 + b_3 > b_1 + b_4 + D$, then choose b_2. P_1 wins.

   P_1 wins if they get b_2 and b_3, which means b_2 and b_4 work as well. This is because $b_2 + b_3 > b_1 + b_4 + D$. After P_1 chooses b_2, P_2 cannot choose both b_3 and b_4 in the same turn. So P_1 wins.

   Example: $(\{3, 7, 8, 9\}, 2)$

B. If $b_2 + b_3 = b_1 + b_4 + D$, then choose b_2. P_1 draws but can win if P_2 makes a mistake.

P_1 loses if they do not choose b_2. If P_1 chooses b_1, then P_2 could choose b_2, limiting P_1 to choosing at most one value again, and P_1 has to choose b_3 or b_4 and loses. If P_1 chooses b_3, P_2 could choose b_2 and b_4 and P_1 loses. If P_1 starts with b_4, P_2 gets all the values remaining. If P_1 stars with b_2, P_2 could choose b_1 and b_4 and because $b_2 + b_3 = b_1 + b_4 + D$ it is a draw, but if P_2 chooses b_3 instead then P_1 wins by choosing b_1 and b_4.

Example: $(\{2, 4, 6, 7\}, 1)$

C. If $b_2 + b_3 < b_1 + b_4 + D$, then consider the following cases:

This means that P_1 cannot choose b_2 or b_3 because P_2 could get b_1 and b_4 in which case P_1 loses. So P_1 should choose b_1 or b_4 depending on the following further conditions.

1. If $b_1 + b_4 > b_2 + b_3 + D$, then consider the following cases:

   a. If $b_4 < b_1 + b_2 + b_3 + D$, then choose b_1. P_1 wins.

   After choosing b_1, P_1 only needs b_4 to win. P_2 choose b_4, but after that P_1 gets all the remaining values in B. Since b_4 cannot win by itself, P_1 wins.

   Example: $(\{3, 7, 8, 9\}, 2)$

   b. If $b_4 > b_1 + b_2 + b_3 + D$, then choose b_4. P_1 wins.

   No matter if P_2 gets all the remaining numbers in B, no sum of the remaining number is greater then or equal to b_4 by itself.

   Example: $(\{4, 7, 8, 30\}, 3)$

   c. If $b_4 = b_1 + b_2 + b_3 + D$, then choose b_4. P_1 draws.

   P_1 is limited to choosing at most one number in the first move. P_1 should not choose any combination of b_1, b_2 or b_3 because P_2 would choose b_4

afterward and P_1 would lose. So P_1's best option is to draw by choosing b_4 first.

Example: $(\{4, 8, 9, 24\}, 3)$

2. If $b_1 + b_4 = b_2 + b_3 + D$, then choose b_1. The outcome for P_1 depends on further conditions.

   If P_1 chooses b_4, then P_2 could chooses b_1, b_2, b_3 in which P_1 loses. So the other viable option for P_1 is to choose b_1, in which case P_2 could choose b_2 and P_1 could choose b_4 and because $b_1 + b_4 = b_2 + b_3 + D$ it is a draw.

   But P_2 could choose b_4 instead, then P_1 would get b_1, b_2 and b_3. The result depends on the following cases:

   a. If $D = b_1$, then the outcome is a draw for P_1. This is because the hypothesis says that $b_1 + b_4 = b_2 + b_3 + D$, and if $D = b_1$, then $b_4 = b_2 + b_3$. By the sequence of choices mentioned above where P_2 could choose b_4 instead, P_1 would end up with b_1, b_2 and b_3 while P_2 would end up with b_4.

      Example: $(\{3, 4, 5, 9\}, 3)$

   b. If $D \neq b_1$, this means that $b_1 > D$, then the outcome is a win for P_1. This is because by the sequence of choices mentioned above where P_2 could choose b_4 instead, P_1 would end up with b_1, b_2 and b_3 while P_2 would end up with b_4. Since $b_1 + b_4 = b_2 + b_3 + D$, but $b_1 > D$, so P_1 would win.

      Example: $(\{2, 3, 6, 8\}, 1)$

3. If $b_1 + b_4 < b_2 + b_3 + D$, then choose b_4. The outcome can still be a win, loss or draw for P_1 depending on further sub cases.

The strategy of choosing b_1 for P_1 does not work anymore. We know that b_2 and b_3 are not viable options either, so the only remaining option is to choose b_4 first.

Now consider whether $D - b_4$, is less than, greater than, or equal to b_2, and apply the results of section 2.2.3 ($D \neq 0$ and $n = 3$.)

II. If $D > b_1$

A. If $b_1 + b_2 + b_3 > b_4 + D$, then choose b_1 and b_2. P_1 wins.

By choosing b_1 and b_2 first, P_1 only needs b_3 to use the inequality $b_1 + b_2 + b_3 > b_4 + D$ to win. Getting b_4 is even better. P_2 cannot choose b_3 and b_4 in one turn, so P_1 wins.

Example: $(\{4, 5, 7, 8\}, 5)$

B. If $b_1 + b_2 + b_3 = b_4 + D$, then choose b_1 and b_2. P_1 draws.

Note that here P_1 cannot win. If P_1 starts with b_4, P_1 loses because P_2 get all the remaining values in B. If P_1 starts with b_2 or b_3, P_2 chooses b_1 and b_4 and P_1 loses. Therefore, P_1's best option is to draw the game.

Example: $(\{1, 2, 3, 4\}, 2)$

C. If $b_1 + b_2 + b_3 < b_4 + D$, then consider the following cases:

P_1 should not choose any combination of b_1, b_2 or b_3 because P_2 could chooses b_4, in which case P_1 loses. Therefore, the only remaining option is to choose b_4 first.

1. If $b_4 > b_1 + b_2 + b_3 + D$, then choose b_4. P_1 wins.

   No matter whether P_2 gets all the remaining values of B after choosing b_4, no combination can surpass b_4 by itself, so P_1 wins.

Example: $(\{3,4,5,17\},4)$

2. If $b_4 = b_1 + b_2 + b_3 + D$, then choose b_4. P_1 draws.

   P_1 cannot choose any combination of b_1, b_2 or b_3 because if P_2 chooses b_4, P_1 loses. Therefore, P_1's best option is to draw by choosing b_4 first.

3. If $b_4 < b_1 + b_2 + b_3 + D$, then choose b_4. The outcome can still be a win, loss or draw for P_1 depending on further sub cases.

   Now consider whether $D - b_4$, is less than, greater than, or equal to b_2 and apply the results of section 2.2.3 ($D \neq 0$ and $n = 3$).

2.3 Proofs of Theorems

Recall we denote by P_1 as the player with the next first move in the given game state that is being discussed and P_2 as the other player. Now our objective is for P_2 to win. In other words, imagine that P_2 is you and P_1 is your opponent.

P_1 cannot control what P_2 is going to give P_1 for the end game. But, P_2 can control what scenarios to give to P_1 to play. This means we want to observe all the cases where "P_1 loses" because those are the cases where P_2 wins.

2.3.1 Proof of Theorem 1

Theorem 1 *If $D = 0$ and $n = 3$ or $n = 4$, then there exist no winning strategy for P_2 because P_1 always have a strategy to either force a win or a draw.*

When $D = 0$ for both $n = 3$ and $n = 4$. There are 5 cases where P_1 can force a win, 3 cases where P_1 can force a draw, and no cases where P_2 can force a win. From sections 2.2.1 and 2.2.2, there is not a single case where it says that "P_1 loses."

2.3.2 Proof of Theorem 2

Theorem 2 *Suppose $D \neq 0$ and $n = 3$. P_2 has a winning strategy* ***if and only if*** *one of the following conditions are met:*

1. $|D| > b_2$, $b_2 + b_3 < b_1 + |D|$, *and* $b_1 + b_2 + b_3 < |D|$,

2. $|D| > b_2$, $b_2 + b_3 < b_1 + |D|$, *and* $b_1 + b_2 \geq |D|$,

3. $|D| \leq b_2$, $b_1 < |D|$ *and* $b_1 + b_3 < b_2 + |D|$,

4. $|D| \leq b_2$, $b_1 > |D|$, $b_1 + b_3 < b_2 + |D|$ *and* $b_3 \leq b_1 + b_2 + |D|$,

5. $|D| \leq b_2$, $b_1 = |D|$, $b_1 + b_2 + |D| > b_3$ *and* $|D| - b_3 > b_1$.

For this proof, the results come from section 2.2.3 using the case-by-case analysis and observing all the cases and the sub cases to determine when P_2 has a winning strategy for when $D \neq 0$ and $n = 3$.

1. If $D > b_2$, there are 2 cases where P_2 can force a win, 2 cases where P_2 can force a draw, and 2 cases where P_1 can force a win.

   When $b_2 + b_3 < b_1 + D$ and $b_1 + b_2 + b_3 < D$ or $b_1 + b_2 \geq D$, there exist a winning strategy for P_2.

2. If $D \leq b_2$, there are 3 cases where P_2 can force a win, 4 cases where P_2 can force a draw, and 5 cases where P_1 can force a win. It can get more specific if we consider the following cases.

   (a) If $b_1 < D$, there is 1 case where P_2 can force a win, 1 case where P_2 can force a draw, and 1 case where P_1 can force a win.

   When $b_1 + b_3 < b_2 + D$, there exist a winning strategy for P_2.

(b) If $b_1 > D$, there is 1 cases where P_2 can force a win, 2 cases where P_2 can force a draw, and 2 cases where P_1 can force a win.

When $b_1 + b_3 < b_2 + D$ and $b_3 < b_1 + b_2 + D$, there exist a winning strategy for P_2.

(c) If $b_1 = D$, there is 1 cases where P_2 can force a win, 1 case where P_2 can force a draw, and 2 cases where P_1 can force a win.

When $b_3 < b_1 + b_2 + D$ and $D - b_3 > b_1$, there exist a winning strategy for P_2.

2.4 Summary

Our goal was to produce all the winning and drawing strategies in the end game when $n = 3$ and $n = 4$, thereby solving the game from those states. Through the results of the research, we were able to understand what to do in all of these scenarios as the player who has the next move. Applying the case-by-case analysis, we found the best moves in all of these situations.

It is still complicated to memorize what to do in each scenario. This means that all these cases are not created to be memorized but rather to serve as a guide to understand and build intuition for the game. They do not culminate in a heuristic strategy but could be further analyzed to potentially produce one.

An inference that can be made from the results is an enhanced comprehension of how the game works which leads to a nearer step finding a solution to Catch-up. Another type of interpretation is the recreational value that it provides, some people might just have fun playing Catch-up. They are interested in finding out more information about the game so they can create better strategies to win their games. Furthermore, This can inspire high school or undergraduate students to enjoy math and motivate them to get involved in research.

3 Future Research

Because of the complex nature of Catch-up, there are many aspects to tackle. A suggestion for future research that seems interesting and useful would involve investigating all the attainable end-game possibilities in Catch-up when there are five numbers remaining $n = 5$. An exciting idea to implement would be applying backward induction to create the cases and results of $n = 5$ by using the results of $n = 3$ and $n = 4$ as foundations.

A more challenging yet intriguing research idea entails the mid-game, since we have the ability to predict the end game outcomes when $n = 3$ and $n = 4$. More research in the mid-game could increase our knowledge of how the game works. Even though it is hard to recognize who is in a winning position early on, there might be ways to have a better understanding of using our results about the end game. Therefore exploring and creating strategies in the mid-game can be beneficial for the transition from mid to end game. This way we can have a greater insight on how the game operates in general.

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