Theory and Applications of Topological Data Analysis

A Thesis Presented in Partial Fulfillment of

the Honors Bachelor's Degree

Eric Connelly

Abstract

The topic of my thesis is the theory and applications of topological data analysis. We will develop tools to analyze the shape of data using ideas from topology. First we discuss metric spaces and then we discuss ideas from topology like continuous functions, homeomorphisms, subspace, product, and quotient topologies, and homotopy equivalence. We introduce the concept of a simplicial complex and the various ways of generating them from data. From there we discuss simplicial homology, an important tool from algebraic topology which we will modify in order to analyze our data. We next introduce the idea of persistence, persistent vector spaces, and the classification of all finitely presented persistent vector spaces. Using the ideas from the chapter on persistence we develop the main tool of the paper: persistent homology, and then introduce the space of barcodes, and then a program called Ripser. Finally, we discuss the application of persistent homology to studying natural image data.

Mathematics Department

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1 Introduction

In this thesis we will be talking about a mathematical tool which helps us approximate the "shape" of data, and then looking at applications of this mathematical tool. Why are we interested in finding the shape of data? Finding the shape of data allows us to create models for the data, and this could help with understanding the data.

Our data sets will be called "point clouds"- also known as finite metric spaces. In this thesis we will be trying to approximate the shape of point clouds. The mathematical tool we will trying to understand is called *persistent homology.* The information gained from persistent homology will give us information in the form of something called a *barcode.* We can see an example of a barcode in the right side of figure 1.

Just as in algebraic topology, using homology we can see the number of holes in a topological space, given a point cloud, persistent homology can tell us the number of holes in the unknown shape which our point cloud approximates. In figure 1, visually we can agree that the point cloud on the left approximately has the shape of a circle. Then through persistent homology we end up with a barcode diagram, as shown on the right side of figure 1. The idea is that the long interval corresponds with the single hole of the point cloud circle, and thus represents a true feature.

We focused on the hole in the circle from figure 1, but we will see that persistent homology captures other information, like connected components and higher dimensional holes.

Persistent homology is just one tool to help find the shape of point clouds, and when combined with other mathematical tools, we can make even better approximations.

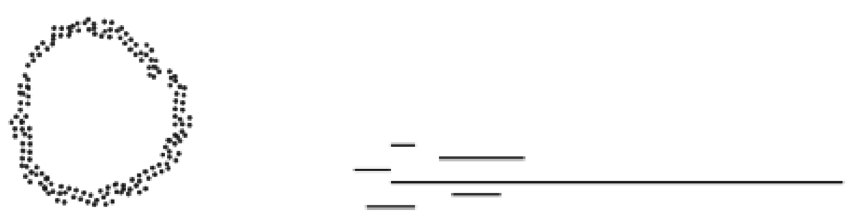

Figure 1: Left: Statistical Circle Right: Barcode generated from Statistical Circle. Source: [2] Topological Data Analysis

In section 2 we introduce the idea of a metric space, provide different examples of metrics, define point clouds, and introduce open sets.

In section 3 we define a topological space and give various examples of topological spaces. We give constructions of more topological spaces such as subspaces, product spaces, and quotient spaces. We introduce topological properties, and then introduce the definiton of homotopy and provide examples.

In section 4 we define simplices, and then using simplices we introduce the definition of a simplicial complex, and then we show how to generate new simplicial complexes from point clouds.

In section 5 we introduce simplicial homology, a tool used to detect holes in simplicial complexes. We then go over examples of simplicial homology.

In section 6 we introduce the idea of persistence. Instead of constructing one simplicial complex from a point cloud, we instead construct a family of simplicial complexes from point clouds. We go on to develop the idea of persistent vector spaces for our family of simplicial complexes.

In section 7 we introduce persistent homology, a tool used to help measure the shape of data. We then introduce barcodes, a visualization of the persistent homology of a point cloud. We then look at the space of barcodes, in order to see how barcodes change with respect to changes in our data. We introduce Ripser, a computer program that can be given point clouds as input and produces barcode diagrams as ouput that represents the persistent homology of our point cloud.

Finally, in section 8 we look at an application of persistent homology to natural image data, that is, we consider the space of natural images (images captured by a digital camera), and then consider the space of image patches, where each image patch is a 3 by 3 grid of pixels which we represent using a vector in $\mathbb{R}^9$. Then after some transformations, we apply persistent homology to our point cloud of image patches.

2 Metric Spaces

Given a set X, we would like to define a distance measure between elements in X. If we are working with points in 3-dimensional space, one such measure would be the Euclidean distance between the two points. Our distance measure does not have to be standard Euclidean distance, and our set doesn't have to be a subset of Euclidean Space. We can define an abstract distance measure on sets and think of it as a dissimilarity measure.

In order to create this distance/ dissimilarity measure, we define a function called a metric, which when given a pair of elements, gives us a measure of how different or far apart the two elements are.

We do this by defining a function

$$d : X \times X \to \mathbb{R},$$

which, when is defined in a way to meet certain conditions, is what we call a metric or distance function on X. The conditons are made explicit in the following definition.

Definition 2.1 (Metric Space)**.** We say the pair (X, d), where X is a set and d is a function $d : X \times X \to \mathbb{R}$, is a metric space if and only if the following conditions hold:

1. For any $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.
2. For any $x, y \in X$, $d(x, y) = d(y, x)$
3. (*Triangle Inequality*) For any $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$

Example 2.2 ($\mathbb{R}$; the real numbers)**.** *Consider the set $\mathbb{R}$ with the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined with*

$$d(x, y) = |x - y|,$$

where the vertical bars "$|\ |$" mean absolute value. $(\mathbb{R}, d)$ *is a metric space.*

We see the the triangle inequality condition is fulfilled by the following:

$$d(x, z) = |x - z| = |x - y + y - z| \leq |x - y| + |y - z| = d(x, y) + d(y, z),$$

where the inequality comes from the typical triangle inequality for the absolute value.

Example 2.3 (($\mathbb{R}^n$,d_2) ; n-dimensional Euclidean space with Euclidean distance)**.** *Consider the set $\mathbb{R}^n$ with $n \geq 2$ and the function*

$$d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined for any points $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ as

$$d\Big((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)\Big) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

The pair $(\mathbb{R}^n, d_2)$ is a metric space. The triangle inequality is fulfilled using the Cauchy-Schwarts inequality.

We can define other metrics on the set $\mathbb{R}^n$. Two examples are the d_1 and d_∞ metrics.

Example 2.4 (($\mathbb{R}^n$,d_1))**.** *Consider the set $\mathbb{R}^n$ with $n \geq 2$ and the function*

$$d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined for any points $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ as

$$d_1\Big((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)\Big) = |y_1 - x_1| + |y_2 - x_2| + ... + |y_n - x_n|$$

The pair $(\mathbb{R}^n, d_1)$ is a metric space.

Example 2.5 (($\mathbb{R}^n$,d_∞))**.** *Consider the set $\mathbb{R}^n$ with $n \geq 2$ and the function*

$$d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

defined for any points $(x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n) \in \mathbb{R}^n$ as

$$d_\infty\Big((x_1, x_2, ..., x_n), (y_1, y_2, ..., y_n)\Big) = max\{|y_1 - x_1|, |y_2 - x_2|, ..., |y_n - x_n|\}$$

The pair $(\mathbb{R}^n, d_\infty)$ is a metric space.

Example 2.6 (Discrete Metric Space). *Given any set X, we can define the metric d_{dis},called the discrete metric, where for any $x, y \in X$, we have*

$$d_{dis}(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Definition 2.7 (Open Ball). An important idea in any metric space (X, d) is the the idea of an open ball. We define an open ball as the set with center x_0 and radius ϵ defined with

$$B(x_0, \epsilon) = \{x \in X \mid d(x_0, x) < \epsilon\}$$

Definition 2.8 (Closed ball). We define a closed ball as the set with center x_0 and radius ϵ defined with

$$B[x_0, \epsilon] = \{x \in X \mid d(x_0, x) \leq \epsilon\}$$

In figure 2 below we show the open balls in $\mathbb{R}^2$ with metrics d_2, d_1, and d_∞.

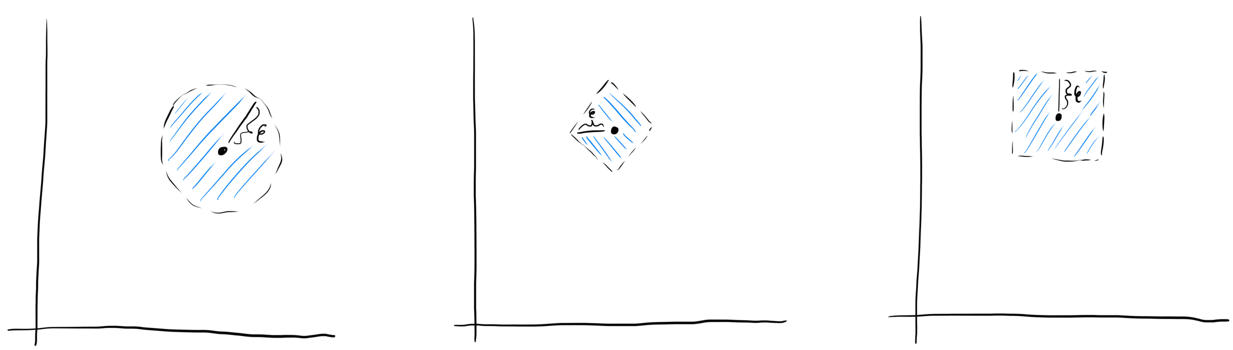

Figure 2: Left: Open Ball Under d_2. Center: Open Ball Under d_1. Right: Open Ball Under d_∞

Definition 2.9 (Open Set in a Metric Space). Given a metric space (X, d), we will say a subset $U \subseteq X$ is open if and only if for any $x \in U$, there exists an $\epsilon > 0$ such that we have $B(x, \epsilon) \subseteq U$.

Some examples open sets in $\mathbb{R}^n$ are open balls, $\mathbb{R}^n$ itself, and the empty set. In $\mathbb{R}^2$, one example of an open set is $\{(x, y) \mid y > 0\}$. In $\mathbb{R}$, the set $\mathbb{Q} \subseteq \mathbb{R}$ is not open, the set $\mathbb{Z} \subseteq \mathbb{R}$ is not open. Closed balls in $\mathbb{R}^n$ are not open.

Definition 2.10 (Point Cloud)**.** Given a finite set $\mathbb{X}$ and a metric on $\mathbb{X}$, we will call $\mathbb{X}$ a finite metric space or a point cloud.

Examples of point clouds are given in figure 3.

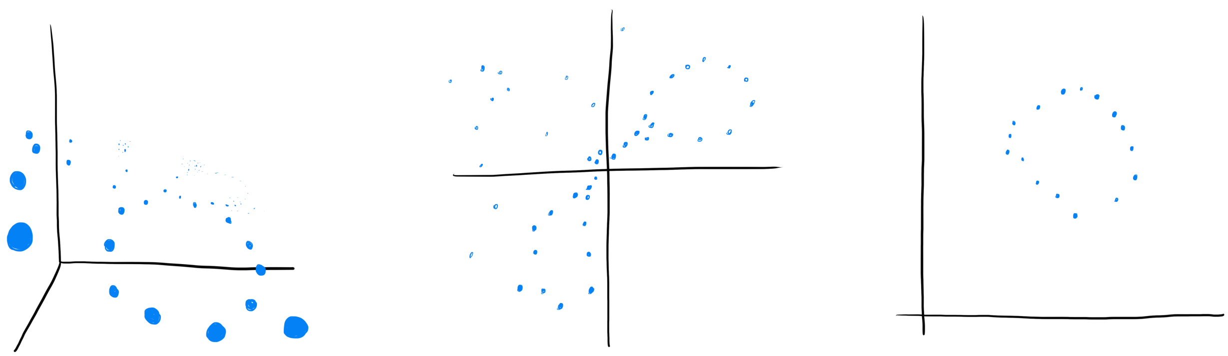

Figure 3: Three examples of point clouds.

Theorem 2.11. *Let X be a finite set and d be a metric on X. Then (X, d) is a discrete metric space.*

3 Topology

We can make the idea of a metric space more abstract by defining closeness using open sets instead of distance. When talking about metric spaces we had a set with a metric, which was a distance/ dissimilarity measure that met certain conditions which told us some information about how pairs of elements lie with respect to each other. We then defined open sets in terms of open balls. This definition depended on a choice of metric. We now want to create a way to define openness on sets, where we won't always have to choose a metric in order to define openness. We will define what it means for a set to be open in a set X by choosing something called a topology. For example we can define a topology on $\mathbb{R}^n$ which gives us the same open sets as defined using a metric.

Definition 3.1 (Topology)**.** Given a set X, let $\mathcal{T}$ be a collection of subsets from X. We say that $\mathcal{T}$ is a topology on X if and only if the following conditions hold:

1. $X \in \mathcal{T}$ and $\varnothing \in \mathcal{T}$.
2. Any arbitrary union of sets from $\mathcal{T}$ is also an element of $\mathcal{T}$.
3. Any finite intersection of sets from $\mathcal{T}$ is also an element of $\mathcal{T}$.

Definition 3.2 (Topological Space)**.** We will call the pair $(X, \mathcal{T})$ a topological space, or a space for short, and we will call the the elements of $\mathcal{T}$ the open sets of X.

A key point here is that we are defining what it means for a subset U of X to be considered an open set: U is open in X precisely when we have $U \in \mathcal{T}$. Once we have a topology $\mathcal{T}$ defined on X, for any set $U \subseteq X$, we have that

$$U \text{ is open in } X \text{ if and only if } U \in \mathcal{T}.$$

Example 3.3 (Metric Topology)**.** *Given a metric space (X, d), Define a topology $\mathcal{T}$ where for any $U \subseteq X$, we have $U \in \mathcal{T}$ if and only if for any $x \in U$, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. This is called the metric topology on X.*

An important fact about the metric topology is that the metric space definition of open sets matches the topological definition of open sets.

Example 3.4 (Discrete Topology). *Given a set X, let the topology $\mathcal{T}$ on X be the power set of X. Then every subset of X is defined to be open. This topology is called the discrete topology. In the discrete topology singletons are open.*

Example 3.5 (Trivial Topology). *Given a set X let the topology $\mathcal{T}$ on X be $\{X, \varnothing\}$. We call this topology the trivial topology.*

Example 3.6 ($\mathbb{R}^n$). *Consider the set $\mathbb{R}^n$ with topology $\mathcal{T}$ where for any $U \subseteq \mathbb{R}^n$, we have $U \in \mathcal{T}$ if and only if for any $x \in U$, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$.*

Like a basis in vector spaces, there is an idea of a basis for a topology, and a topology generated by a basis. Basis open sets are very useful in describing many topological spaces and working with them.

Definition 3.7 (Topological Basis). Given a topological space X, if we can find a set $\mathcal{B}$ which has the following properties,

1.) For any $x \in X$, there is a set $B \in \mathcal{B}$ such that we have $x \in B$ and

2.) For any $x \in X$, if $x \in B_1$ and $x \in B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there exists a set $B_3 \in \mathcal{B}$ such that we have $B_3 \subseteq B_1 \cap B_2 \in \mathcal{B}$,

then we call $\mathcal{B}$ a basis for the topology on X.

We note that $U \in \mathcal{T}$ if and only if for any $x \in U$, there exists a $B \in \mathcal{B}$ containing the element x such that $B \subseteq U$.

Example 3.8 (Basis for $\mathbb{R}$ with standard topology). *The set of open intervals forms a basis for* $\mathbb{R}$.

We can see a union of basis elements forming one longer interval in figure 4.

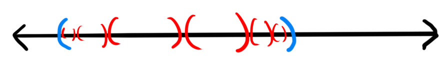

Figure 4: Using the basis for $\mathbb{R}$ to to make open set.

Example 3.9 (Basis for $\mathbb{R}^n$ with standard topology)**.** *The set of open balls forms a basis for* $\mathbb{R}^n$

We can see a union of open balls forming an open set in figure 5.

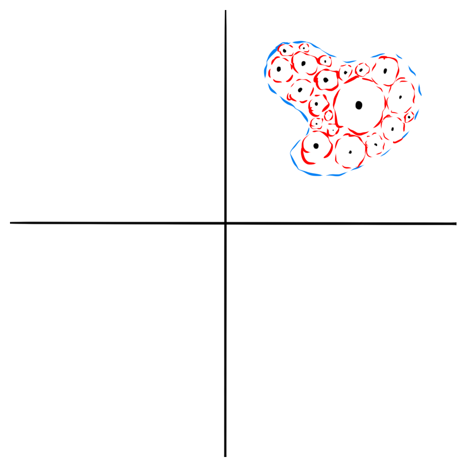

Figure 5: Using the basis for $\mathbb{R}^2$ to to make open set.

The sets of open balls defined using d_1, d_2, d_∞ all form a basis for the standard topology on the set $\mathbb{R}^n$. Using these open balls we can see that all these metrics induce the same topology on $\mathbb{R}^n$. We can see this in figure 6.

Definition 3.10 (Topology generated by a Basis)**.** Given a collection $\mathcal{B}$ of subsets from a set X, if $\mathcal{B}$ satisfies all of the conditions from definition 3.7, then there is a topology $\mathcal{T}_\mathcal{B}$ on X defined as follows

$$\mathcal{T}_\mathcal{B} = \left\{ \bigcup_{\alpha \in A} B_\alpha \mid A \text{ is any index set and } B_\alpha \in \mathcal{B} \text{ for all } \alpha \in A \right\}$$

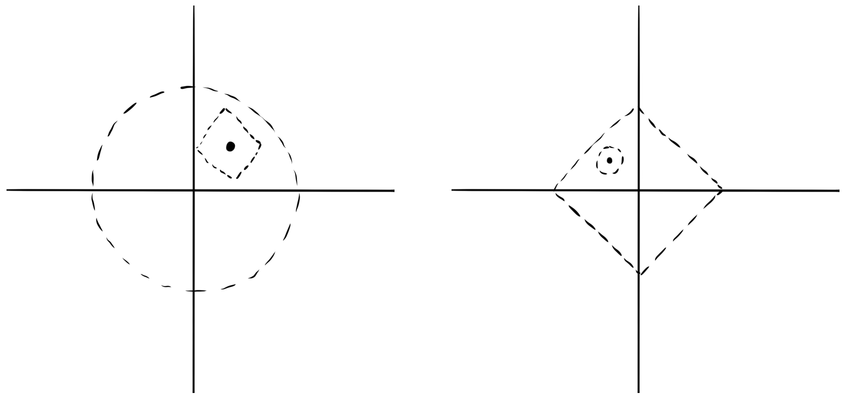

Figure 6: Open balls in d_2-metric are open in d_1-metric and open balls in d_1-metric are open in d_2-metric.

$\mathcal{T}_{\mathcal{B}}$ is called the topology generated by $\mathcal{B}$ and $\mathcal{B}$ is a basis for $\mathcal{T}_{\mathcal{B}}$.

3.1 Generating New Topologies from Old

We describe three constructions to define new topologies from given topological spaces - subspace topology, product topology, and quotient topology.

Definition 3.11 (Subspace Topology)**.** Given a topological space X and a subset $A \subseteq X$, we can define a topology on A such that for any $V \subseteq A$, V is open in A if and only if there exists $U \subseteq X$ such that $V = U \cap A$. This is called the subspace topology on A.

Figure 7 shows an example of a subspace of $\mathbb{R}^2$.

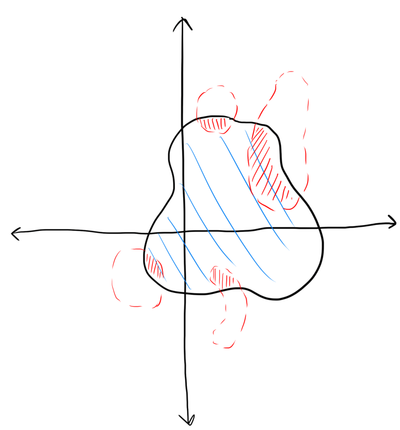

Figure 7: Area enclosed by red dotted line: Open set in X. Area filled in red: Open set in subspace $A \subseteq X$

Example 3.12. *$\mathbb{Z} \subseteq \mathbb{R}$ with the subspace topology is the same as giving $\mathbb{Z}$ the discrete topology.*

Example 3.13 (A point cloud $\mathbb{X} \subseteq \mathbb{R}^n$)**.** *Given a point cloud $\mathbb{X} \subseteq \mathbb{R}^n$, we can give $\mathbb{X}$ the subspace topology induced from $\mathbb{R}^n$. Seeing that $\mathbb{X}$ is a finite set, for any $x \in \mathbb{X}$, there exists an $\epsilon > 0$ such that $B(x, \epsilon) \cap \{x\} = \{x\}$, and so $\{x\}$ is open in the subspace $\mathbb{X}$ by the definition of the subspace topology. Notice this gives our point cloud $\mathbb{X}$ the discrete topology.*

Example 3.14 (The n-sphere S^n)**.** *Consider the set*

$$S^n = \{(x_1, x_2, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + x_2^2 + ... + x_{n+1}^2 = 1\}.$$

We can give $S^n \subseteq \mathbb{R}^{n+1}$ the subspace topology by intersecting open balls from $\mathbb{R}^{n+1}$ with S^n. For example, open balls with small radii from $\mathbb{R}^2$ intersect with S^1 and give us "open arcs" as open sets for our subspace S^1. See figure 8.

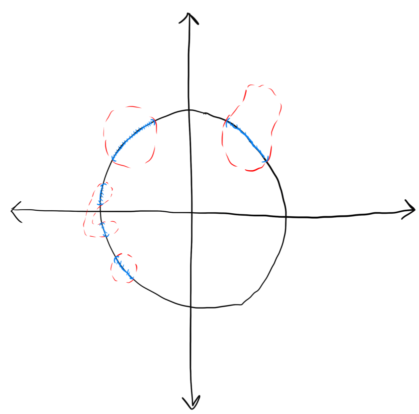

Figure 8: S^1 as a subspace of $\mathbb{R}^2$

Given sets X and Y, we can take the Cartesian product

$$X \times Y = \{(x, y) \mid x \in X \text{ and } y \in Y\},$$

but what if X and Y are topological spaces? How do we define the topology on $X \times Y$ using the topologies from X and Y?

Definition 3.15 (Product Topology)**.** We will define the product topology on the set $X \times Y$ as the topology generated by the basis

$$\{U \times V \mid U \text{ open in } X \text{ and } V \text{ open in } Y\}$$

Example 3.16 ($\mathbb{R}^n$)**.** *Consider* $\mathbb{R}$ *with the standard topology induced by the metric*

$$d(x, y) = |x - y|.$$

We can take the Cartesian product $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$*, and we see that that the basis for our product is the collection of open balls under the* d_∞ *and hence a basis for the standard topology on* $\mathbb{R}^2$*. By induction the standard topology on* $\mathbb{R}^n$ *can be induced by taking the product* $\mathbb{R}^{n-1} \times \mathbb{R}$*.*

Theorem 3.17. *Suppose we have subsets* $A \subseteq X$ *and* $B \subseteq Y$*, then the subspace topology for* $A \times B \subseteq X \times Y$ *is the same as the product topology on* $A \times B$*, with the subspace topology on* A *and* B*.*

Example 3.18 (Torus)**.** *Consider the set* S^1 *with the subspace topology induced from* $\mathbb{R}^2$ *with the standard topology. We can generate a torus by taking the product* $S^1 \times S^1$*. See figure 9.*

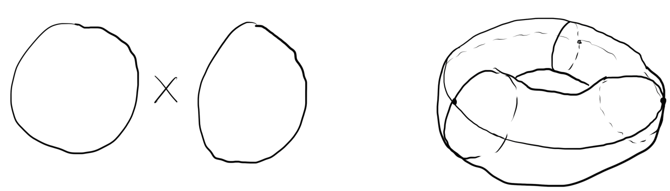

Figure 9: Torus as a product. Note: the torus is also a subspace of $\mathbb{R}^4$.

Definition 3.19 (Quotient Space)**.** Given a topological space and an equivalence relation $\sim$ on X and $[X] = X/\sim$, the set of equivalence classes, and a function $q : X \to [X]$ defined as $q(x) = [x] \in [X]$. We can define a topology on $[X]$ as follows:

For any subset $U \subseteq [X]$, U is open in $[X]$ if $q^{-1}(U)$ is open X.

This is called a quotient topology and q is called a quotient map.

We can capture the idea of gluing using quotient spaces. When we have two different elements x and y related to each other under the equivalence relation which we are quotienting with, we can think of this as gluing x and y together. We will want to show later that the result of our gluing process results in familiar spaces.

3.2 Continuous Functions

Definition 3.20 (Continuous Functions)**.** We want functions which preserve the topological structure of a space. Given a function

$$f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$$

between topological spaces, we will say f is is continuous if and only if for any open set U in Y, $f^{-1}(U)$ is open in X.

For example, all of the functions $f : \mathbb{R}^n \to \mathbb{R}^m$ which are continuous with the epsilon delta definition of continuity from calculus is continuous under the standard topology.

Example 3.21. *Let X be a topological space. Consider a subspace $A \subseteq X$. The inclusion function $i : A \to X$ is continuous.*

Example 3.22. *Consider the product space $X_1 \times X_2$, where X_1 and X_2 are topological spaces. The function $p_i : X_1 \times X_2 \to X_i$ with $i \in \{1, 2\}$ and defined by $p(x_1, x_2) = x_i$ is continuous.*

Example 3.23. *The quotient map $q : X \to [X]$ is continuous where $[X]$ has the quotient topology.*

After gluing elements together, in order to show our resulting quotient space actually resembles a more familiar space, we have to show that our quotient space is "the same" as our desired familiar space. We capture this sameness with the following definition:

Definition 3.24 (Homeomorphism)**.** Given a function

$$f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$$

between topological spaces, we will say f is a homeomorphism if and only if f is a continuous bijection with a continuous inverse.

Example 3.25 (Gluing the ends of a String Together)**.** *The circle S^1 is homeomorphic to an interval with ends identified. See figure 10.*

Given the unit interval $I = [0,1]$, consider the equivalence relation $\sim$ where $x \sim y$ if and only if $x = y$ or $x, y \in \{0,1\}$. This is like gluing 0 and 1 together to get a loop. It turns out we can find a homeomorphism between $I/\sim$ and $S^1 \subseteq \mathbb{R}^2$. Namely, $f : [I] \to S^1$ defined with

$$f([x]) = (cos(2\pi x), sin(2\pi x))$$

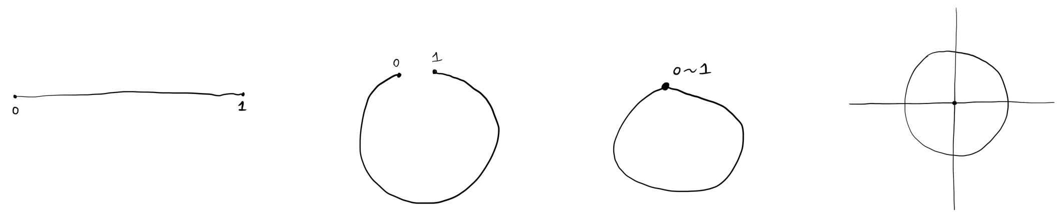

Figure 10: Circle.

We can take the unit square $[0,1] \times [0,1]$ and glue its edges in different ways to get different quotient spaces. We can use pictures to capture the idea of gluing. Arrows drawn on the edges of the square tell us how to glue. We want in our finished product after gluing to have arrows of the same type overlapping each other.

Example 3.26. *Torus; Gluing ends of a tube together. Torus is homeomorphic to a quotient space of a square with its sides glued together . See figure 11.*

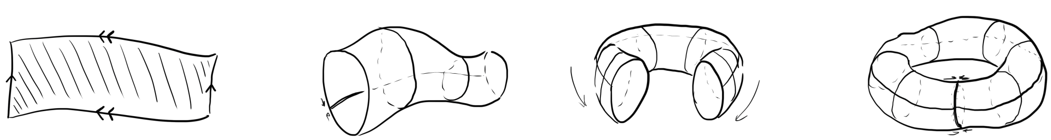

Figure 11: Torus as an identification on a square.

Example 3.27 (Mobius Band)**.** *See figure 12.*

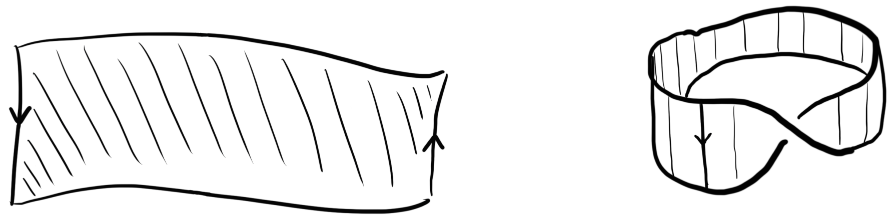

Figure 12: A Mobius band

3.3 Topological Properties

Definition 3.28 (Topological Invariant under Homeomorphism)**.** Given topological spaces X and Y, a property of a space X is said to be a topological property if whenever Y is homeomorphic to X, then the space Y also has the property.

Definition 3.29 (Path)**.** Given a space X, a path in X is a continuous function

$$\lambda : [0, 1] \to X$$

Definition 3.30 (Path-Connected)**.** Given a space X, we will say X is path connected if and only if for any $x_0, x_1 \in X$, there is a path $\lambda : [0, 1] \to X$ with $\lambda(0) = x_0$ and $\lambda(1) = x_1$.

For example, $\mathbb{R}^n$ is path connected; there is a straight line path between any two points. See figure 13 for more examples.

Theorem 3.31. *Given two path-connected spaces X and Y, the product $X \times Y$ is also path-connected.*

Theorem 3.32. *Given a path-connected space X and a continuous function $f : X \to Y$, then the image of X in Y, $f(Y)$, is path-connected.*

Now we have the result:

Theorem 3.33. *Path-connectedness is a topologcal invariant under homeomorphism.*

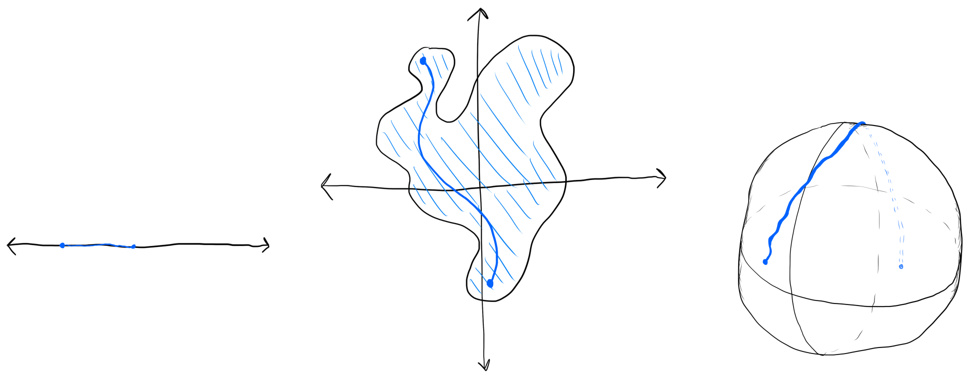

Figure 13: Left: $\mathbb{R}$ is path connected. Center: A path connected subset of $\mathbb{R}^2$. Right: S^2 is path connected.

3.4 Homotopy

Homeomorphism is one type of equivalence of topological spaces - in many instances we need a weaker type of equivalence where we allow deformation without changing the intrinsic topological nature. This is called homotopy equivalence.

Definition 3.34 (Homotopic Functions)**.** Given continuous functions $f : X \to Y$ and $g : X \to Y$, a homotopy H between f and g is a continuous function

$$H : [0,1] \times X \to Y$$

where $H(0, x) = f(x)$ and $H(1, x) = g(x)$.

If there exists a homotopy between f and g, we will say f and g are homotopic, and we will write $f \simeq g$.

Definition 3.35 (Homotopy Equivalence)**.** Given topological spaces X and Y, we will say X and Y are homotopy equivalent if and only if there exists continuous functions $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_Y$. If two topological spaces X, Y are homotopy equivalent, we will write $X \simeq Y$.

Example 3.36 ($\mathbb{R}^n$ and a point)**.** *Consider $\mathbb{R}^n$ and a point $p \in \mathbb{R}^n$. Consider $f : \mathbb{R}^n \to \{p\}$ and $g : \{p\} \to \mathbb{R}^n$. with $f(x) = p$ for all $x \in \mathbb{R}^n$ and $g(x) = p \in \mathbb{R}^n$.*

Then we have the homotopy $H_1 : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ with

$$H_1(x,t) = g \circ f(x)(1-t) + Id_{\mathbb{R}^n} t$$

and the homotopy $H_1 : \{p\} \times [0,1] \to \{p\}$ with

$$H_2(x,t) = f \circ g(x)(1-t) + Id_{\{p\}} t$$

This means we have $g \circ f(x) \simeq Id_{\mathbb{R}^n}$ and $f \circ g \simeq Id_{\{p\}}$. This means that $\mathbb{R}^n$ and a point are homotopy equivalent.

Definition 3.37 (Contractible)**.** If a topological space X is homotopy equivalent to the one point space $\{p\}$, we say X is contractible.

Any convex subset of $\mathbb{R}^n$ is contractible. For example, open balls and closed balls in $\mathbb{R}^n$. See figure 14.

Example 3.38 (A disk and a point)**.**

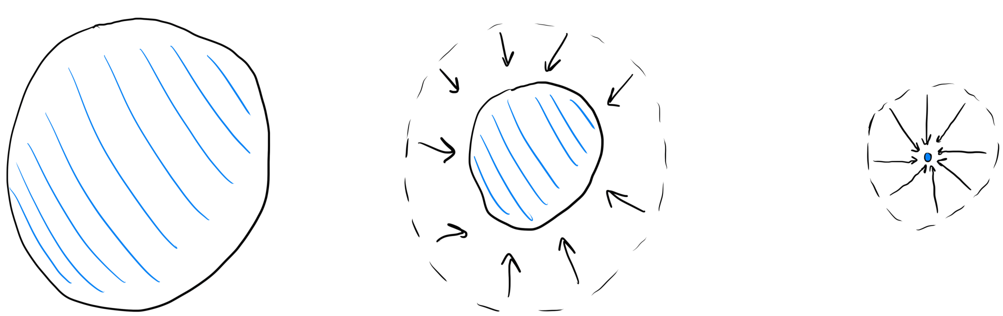

Figure 14: Disk deforming to a point.

Example 3.39 ($\mathbb{R}^2 - \{0\}$ and S^1)**.** *See figure 15.*

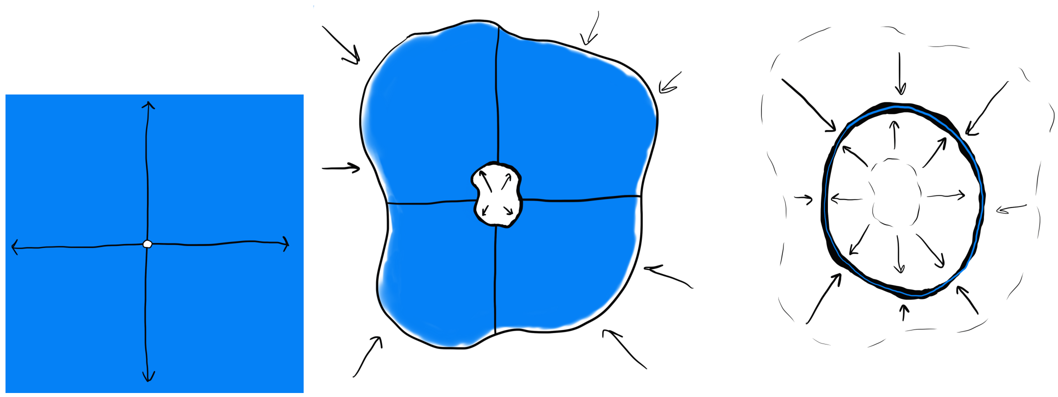

Figure 15: Punctured plane deforming to a circle.

Example 3.40. $\mathbb{R}^n$ *and* S^{n-1} *are homotopy equivalent.*

Example 3.41. $X \times [0,1]$ *and* X *are homotopy equivalent.*

If X in example 3.41 was S^1, we can think of $X \times [0,1]$ as a hollow cylinder with no top or bottom, and then we see that we can squish it down to just a circle.

4 Simplicial Complexes

We want to be able to build topological spaces using simple building blocks and by identifications, that is, using equivalnece relations and giving the resulting spaces the quotient topology. One way to do this is through constructing simplicial complexes. We build these spaces using simplices, which are generalizations of triangles.

4.1 n-simplex

Consider a set of points $V = \{v_0, v_1, ..., v_n\} \subseteq \mathbb{R}^{n+1}$ with the property that the set vectors $\{v_1 - v_0, v_2 - v_0, ..., v_n - v_0\}$ are linearly independent. We will say these vectors are in general position.

We can define the n-simplex $\Delta^n[V] \subseteq \mathbb{R}^{n+1}$ generated from the vertex set V as the set

$$\left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \in [0,1] \right\} \subseteq \mathbb{R}^{n+1}$$

Given a simplex $\Delta[S]$, we note $\Delta[S]$ is completely defined by its vertex set S.

Example 4.1 ($\Delta^0, \Delta^1, \Delta^2$)**.** *Consider a vertex set $V = \{v_0, v_1, ..., v_n\} \subseteq \mathbb{R}^{n+1}$in general position. Now, suppose V is the set of standard basis vectors for $\mathbb{R}^{n+1}$. Then we write $\Delta^n[V]$ as just Δ^n. See figure 16.*

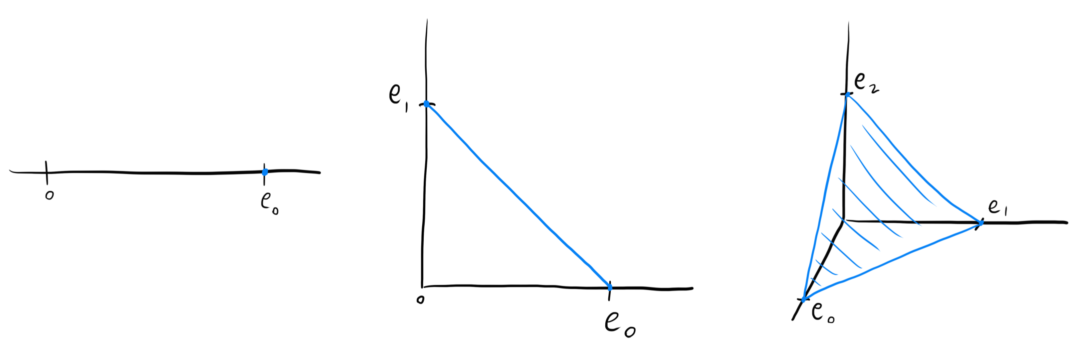

Figure 16: Left:Δ^0, Center: Δ^1, Right:Δ^2.

We can now begin building topological spaces out of simplices.

Definition 4.2 (Geometric Simplicial Complex)**.** Given a finite collection K of simplices in $\mathbb{R}^N$, K forms a simplicial complex if and only if

1. For any n-simplex $S \in K$, where $S \subseteq \mathbb{R}^N$ is some vertex set, any face of S is also in K.

2. For any S and T, we either have $\Delta^n[S] \cap \Delta^m[T] = \Delta^p[U]$ for some $\Delta^p[U] \in K$, or $\Delta^n[S] \cap \Delta^m[T] = \varnothing$.

The dimension of K is the dimension of the n-simplex in K with the highest dimension.

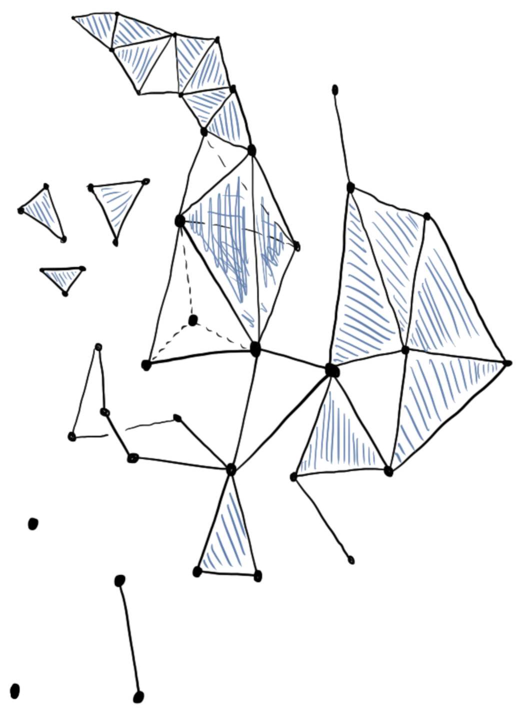

Figure 17: Example of a simplicial complex.

Definition 4.3 (Turning K into a Topological Space)**.** A simplicial complex K determines a topological space $|K| \subseteq \mathbb{R}^N$ called the realization of K. We define $|K|$ as

$$\bigcup_{S \in K} \Delta^n[S]$$

We can turn $|K|$ into a topological space by giving it the subspace topology induced from $\mathbb{R}^N$.

Example 4.4 (Triangle with no interior)**.** *Consider the vertex set* $V = \{v_0, v_1, v_2,\} \subseteq \mathbb{R}^3$ *with the* *collection*

$$K = \Big\{\{v_0\}, \{v_1\}, \{v_2\}, \{v_0, v_1\}\{v_1, v_2\}, \{v_2, v_0\}\Big\}.$$

If we wanted to add the interior to our triangle we would have to have $\{v_0, v_1, v_2, \}$ *as an element of* K*.*

Example 4.5 (Graph)**.** *We can take any set of points in* $\mathbb{R}^2$ *determine a 1 dimensional simplicial complex by drawing edges between points.*

Definition 4.6 (Abstract Simplicial Complexes)**.** Consider a finite collection of finite sets K, where for any set $\tau \in K$, and for any set $\sigma \subseteq \tau$, we have $\sigma \in K$. We will call any $\tau \in K$ a (abstract)simplex, and we will call K a (abstract) simplicial complex. for any $\sigma, \tau \in K$, we will say τ is a face of σ if and only if $\tau \subseteq \sigma$.

We can recover a geometric realization $|K|$ from an abstract simplicial complex by embedding the 0-simplices of K into $\mathbb{R}^N$ as vertices, where N is large enough for our the vertices to be in general position.

Definition 4.7 (Triangulable Spaces)**.** Let K abstract simplicial complex and X be a topological space. X is *triangulable* if X is homeomorphic to the geometric realization of K or, that is, $X \cong |K|$.

4.2 Generating Simpilicial Complexes from Point Clouds

Given a point cloud $\mathbb{X}$, we want to generate simplicial complexes. Below we give 3 constructions.

Definition 4.8 (Cech Complex of a point cloud)**.** Given a point cloud $\mathbb{X}$ we can generate the Cech complex, of $\mathbb{X}$ at scale ϵ, denoted $C(\mathbb{X})_\epsilon$, where any set $\{x_0, x_1, ...x_n\} \subseteq \mathbb{X}$ generates an n-simplex simplex in $C(\mathbb{X})_\epsilon$ if and only if

$$\bigcap_{i=0}^{n} B(x_i, \epsilon) \neq \varnothing$$

Definition 4.9 (Rips Complex of a point cloud)**.** Given a point cloud $\mathbb{X}$ we can generate the Rips complex, of $\mathbb{X}$ at scale ϵ, denoted $R(\mathbb{X})_\epsilon$, where any set $\{x_0, x_1, ...x_n\} \subseteq \mathbb{X}$ generates an n-simplex in $R(\mathbb{X})_\epsilon$ if and only if for any i, j with $0 \leq i, j \leq n$,

$$B(x_i, \epsilon) \cap B(x_j, \epsilon) \neq \varnothing.$$

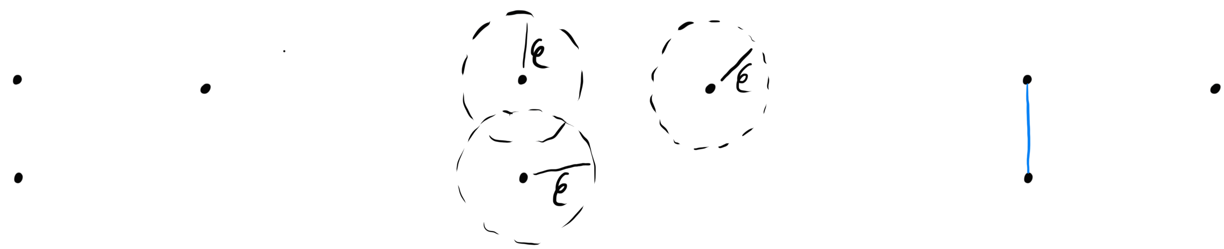

Figure 18: Example of a Cech complex

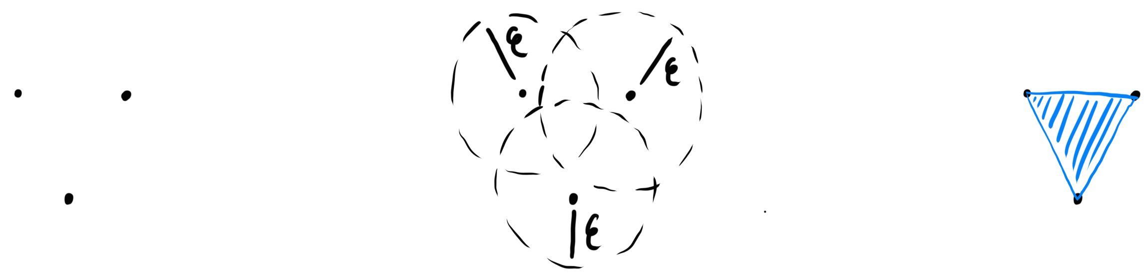

Figure 19: Another example of a Cech complex

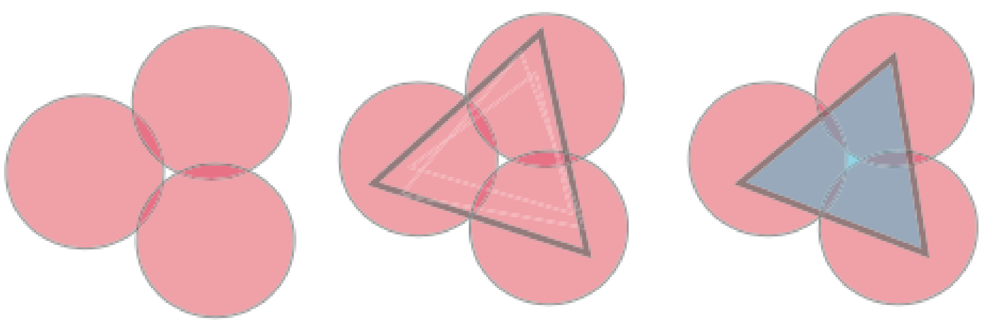

Figure 20: Cech complex v.s. Rips complex. Center: Cech complex. Right: Rips complex. Source: Topolgy and Data [3]

Definition 4.10 (Landmark Points)**.** Consider a point cloud $\mathbb{X} \subseteq \mathbb{R}^n$. We will take a set of points L from $\mathbb{R}^n$ and call it the set of landmark points.

Definition 4.11 (Strong Witness)**.** A point $x \in \mathbb{X}$ is a strong witness for a set $\sigma = \{s_0, s_1, ..., s_k\} \subseteq L$ if we have

$$|x - s_i| \leq |x - \xi|$$

for all $\xi \in \mathbb{X}$.

Definition 4.12 (Weak Witness)**.** A point $x \in \mathbb{X}$ is a weak witness for a set $\sigma = \{s_0, s_1, ..., s_k\} \subseteq L$ if we have

$$|x - s_i| \leq |x - \xi|$$

for all $\xi \in \mathbb{X} - \sigma$.

Definition 4.13 (Strict Witness Complex)**.** We define a witness complex as the simplicial complex $W(L, \infty)$ with vertex set L where a set $\sigma = \{l_0, l_1, ..., l_k\} \subseteq L$ generates a k-simplex in $W(L, \infty)$ if

1. For any $\tau \subseteq \sigma$, we have $\tau \in W(L, \infty)$.
2. There exists $x \in \mathbb{X}$ such that $l_0, l_1, ..., l_k$ are the $k + 1$ nearest neighbors of x in L.

Definition 4.14 (Lazy Witness Complex)**.** We define the lazy witness complex $W_1(L, \mathbb{X})$ as the simplicial complex with the same 1-skeleton as $W_\infty(L, \mathbb{X})$, and with the condition that we have $\sigma \in W_1(L, \mathbb{X})$ if every edge of σ is in the 1-skeleton of $W_\infty(L, \mathbb{X})$.

5 Simplicial Homology

Given a simplicial complex X, we can consider the free vector space over the field $\mathbb{R}$, generated on the set of k-simplices, denoted $C_k(X;\mathbb{R})$, and we can call them the k-chain vector space , or just a chain vector space. If our simplicial complex X has dimension p, then we have $p+1$ vector spaces of the form $C_k(X;\mathbb{R})$.

We can now think about linear transformations between the different chain vector spaces.

One such linear transformation is the boundary linear transformation

$$\partial_{k+1} : C_{k+1}(X;\mathbb{R}) \to C_k(X;\mathbb{R}),$$

which is defined using the basis element $(v_0v_1...v_{k+1})$ of $C_{k+1}(X;\mathbb{R})$, and defined by the evaluation:

$$\partial_{k+1}\Big((v_0v_1...v_{k+1})\Big) = \sum_{i=0}^{k+1}(-1)^i(v_0v_1...v_{k+1})/(v_i)$$

where $(v_0v_1...v_{k+1})/(v_i)$ is the k simplex generated by removing the i-th vertex in $(v_0v_1...v_{k+1})$.See figure 21.

Two very important properties of ∂ is that it is a linear transformation and that $\partial \circ \partial = \partial^2 = 0_{map}$.

Theorem 5.1. $\partial^2 = 0_{map}$

Proof. Consider an n+1-simplex (a basis element)

$$\sigma = \{v_0, v_1, ..., v_{n+1}\} \in C_{k+1}(X;\mathbb{R}).$$

Then we have

$$\begin{aligned}
\partial^2(\sigma) &= \partial_n \circ \partial_{n+1}(\sigma) \\
&= \partial_n \circ \partial_{n+1}(\{v_0, v_1, ..., v_{n+1}\}) \\
&= \partial_n \left(\sum_{i=0}^{n+1} (-1)^i (v_0 v_1 ... v_{n+1})/(v_i) \right) \\
&= \sum_{i=0}^{n+1} (-1)^i \partial_n \Big((v_0 v_1 ... v_{n+1})/(v_i) \Big) \\
&= \sum_{i=0}^{n+1} \left((-1)^i \sum_{j=0}^{n} (-1)^j (v_0 v_1 ... v_n)/(v_j) \right)
\end{aligned}$$

Then we get every simplex in our sum appearing twice with opposite signs and so our sum is equal to zero. We can see this by the following argument: We want to remove two vectors from an n+1-simplex so that we end up with an n-1 simplex. The order in which we remove the two vectors matters. For any vertices v_i,v_j, if we have $i < j$, then v_j will be the $j-1$th vertex after removing v_i, while removing v_i first will leave v_j as the jth vertex in the new simplex. This causes us to have every simplex to appear twice in our sum but with opposite signs, causing each term in our sum to be cancelled out by another term.

□

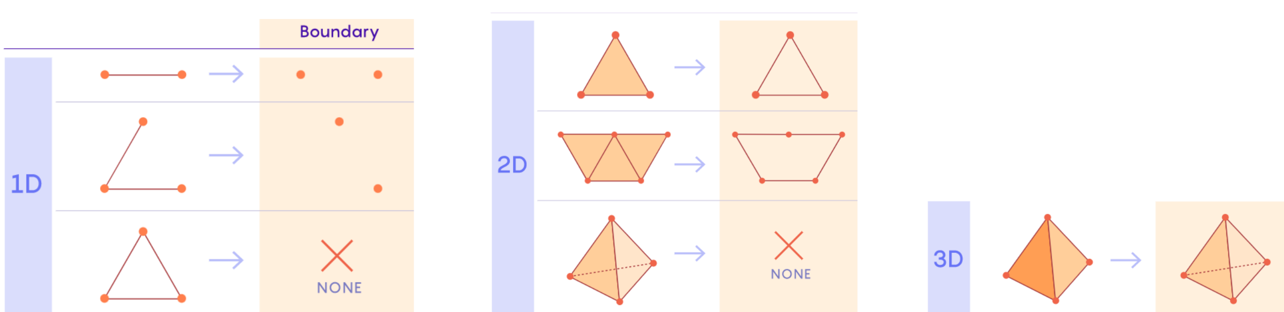

Figure 21: Examples of boundaries. Source: Quanta Article: How Mathematicians Use Homology to Make Sense of Topology.[5]

In order to define the kth-homology vector space, we first need to define the k-cycle and k-boundary subspaces.

Definition 5.2. We define the k-cycles as the k-chains with their boundary equal to zero. We define k-boundaries as the k-chains who lie in the image of the boundary operator ∂_{k+1}, that is, we define the k-cycles as

$$Z_k(X) = \{z \in C_k(X; \mathbb{R}) \mid z \in ker\partial_k\}$$

and we define the k-boundaries as

$$B_k(X) = \{z \in C_k(X; \mathbb{R}) \mid z \in Im(\partial_{k+1})\}$$

Note: We define all 0-simplices as 0-cycles.

Finally, we define the k-th homology vector space as the quotient vector space

$$H_k(X) = \frac{Z_k(X)}{B_k(X)}$$

The k-th homology leaves us with the k-cycles which are not boundaries of linear combinations of $k+1$-simplices. If we think of a triangle with no interior, we can think of the sum of edges as an 1-cycle, while if we take a triangle with an interior, we can think of the edges as being a boundary of a 2-simplex.

We now have two important theorems.

Theorem 5.3 (Simplicial Homology is a Homotopy invariant)**.** *Given two topological spaces X and Y, suppose there exists simplicial complexes K_1 and K_2 such that $X = |K_1|$ and $Y = |K_2|$. If X is homotopy equivalent to Y,then we have $H_i(K_1) = H_i(K_2)$ for all $i \geq 0$.*

Example 5.4 (Homology of a point)**.** *Consider the simplicial complex $X = \{v_0\}$; the one point complex.*

Since all 0-simplices are cycles, we have $Z_0(X) = C_0(X) = span\{v_0\}$, which implies

$$dim(Z_0(X)) = 1$$

Next, since there are no 1-simplices, we have

$$B_0(X) = Im(\partial_1) = \partial_1\{0\}) = 0,$$

which implies that $dim(B_0(X)) = 0$. Now,

$$dim(H_0(X)) = dim(\frac{Z_0(X)}{B_0(X)}) = dim(Z_0(X)) - dim(B_0(X)) = 1 - 0 = 1$$

This means that we have $H_0(X) \cong \mathbb{R}$.

Since $C_i(X) = 0$ for all $i \geq 2$, we must have

$$dim(H_i(X)) = dim(\frac{Z_i(X)}{B_i(X)}) = dim(Z_0(X)) - dim(B_0(X)) = 0 - 0 = 0,$$

which implies that $H_i(X) \cong 0$ for all $i \geq 2$.

To summarise, we have:

$$H_i(X) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}$$

Example 5.5 (Triangle with no Interior)**.** *Consider the "triangle with no interior" simplicial complex*

$$X = \{(v_0), (v_1), (v_2), (v_0v_1), (v_1v_2), (v_0v_2)\}$$

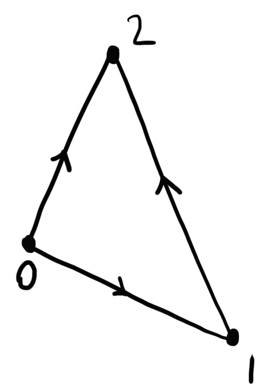

Figure 22: Triangle with no interior.

We first compute $H_0(X)$. Since all 0-simplices are cycles, we see that $Z_0(X) = C_0(X)$. Now, we have

$$\begin{aligned}\partial\Big((v_0v_1)+(v_1v_2)-(v_0v_2)\Big) &= \partial((v_0v_1))+\partial((v_1v_2))-\partial((v_0v_2))\\ &= v_1-v_0+v_2-v_1+v_0-v_2\\ &= 0\end{aligned}$$

but this means that $\partial((v_0v_1)), \partial((v_1v_2)), \partial((v_0v_2))$ are not linearly independent, which means that

$$dim(Im(\partial_{k+1})) = dim(B_0(X)) < 3$$

Now, we prove $\partial((v_0v_1)), \partial((v_1v_2))$ are linearly independent.

Proof. Suppose that there are $a_1, a_2 \in \mathbb{R}$, with a_1, a_2 not both equal to 0,where

$$\begin{aligned}a_1\partial((v_0v_1))+a_2\partial((v_1v_2)) &= a_1(v_1-v_0)+a_2(v_2-v_1)\\ &= a_1v_1-a_1v_0+a_2v_2-a_2v_1\\ &= 0\end{aligned}$$

with the last two lines implying

$$\begin{aligned}(a_1-a_2)v_1 &=0\\ (-a_1)v_0 &=0\\ (a_2)v_2 &=0\end{aligned}$$

But then for all three lines to be true, we need a_1, a_2 to both be equal to 0. A contradiction! □

So now we know that $\partial((v_0v_1)), \partial((v_1v_2))$ are linearly independent, and since $dim(B_0(X)) < 3$, it must be that $\partial((v_0v_1)), \partial((v_1v_2))$ form a basis for $B_0(X)$

$\implies dim(B_0(X)) = 2$.

so then we have

$$dim(H_0(X)) = dim(\frac{Z_0(X)}{B_0(X)}) = dim(Z_0(X)) - dim(B_0(X)) = 3 - 2 = 1$$

$\implies H_0(X) = \dfrac{Z_0(X)}{B_0(X)} \cong \mathbb{R}$

So our 0-th homology vector space is isomorphic to $\mathbb{R}$. We can interpret the dimension of $H_0(X)$ as the number of connected components in our our simplicial complex.

Now we compute $H_1(X)$. First, we find the dimension of $Z_1(X)$. By the rank- nullity theorem, we have that $dim(Im(\partial_1)) + dim(ker\ \partial_1) = dim(C_1(X))$, but we already know $dim(Im(\partial_1))$ and $dim(C_1(X))$, so we can write

$$\begin{aligned}
\dim(Z_1(X)) &= dim(ker\ \partial_1) \\
&= dim(C_1(X)) - dim(Im(\partial_1)) \\
&= 3 - 2 \\
&= 1
\end{aligned}$$

Now all we need is to find the dimension of $B_1(X) = Im(\partial_2)$, but there are no 2- simplices, and just by the property of ∂ being a linear transformation, we know $\partial_2(\{0\}) = 0$, so we can write $dim(B_1(X)) = 0$.

Now, we see

$$\begin{aligned}
dim(H_1(X)) &= dim(\frac{Z_1(X)}{B_1(X)}) \\
&= dim(Z_1(X)) - dim(B_1(X) \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

Which implies

$$H_1(X) = \frac{Z_1(X)}{B_1(X)} \cong \mathbb{R}$$

This is interpreted as saying our simplicial complex has one hole, which matches what we see when we look at a triangle with no interior.

Finally, we see

$$H_2(X) = \frac{Z_2(X)}{B_2(X)} = \frac{\{0\}}{\{0\}} \cong 0$$

and similarly $H_k(X) \cong 0$ for all $k \geq 2$.

To summarise, we have:

$$H_i(X) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ \mathbb{R} & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

Example 5.6 (Triangle with Interior)**.** *Consider the simplicial complex*

$$X = \{(v_0), (v_1), (v_2), (v_0v_1), (v_1v_2), (v_0v_2), (v_0v_1v_2)\}$$

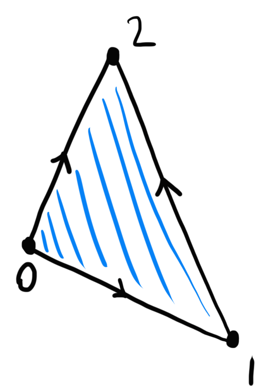

Figure 23: Triangle with interior.

For this example, nothing has changed except now we have a 2-simplex added to our simplicial complex(our triangle is filled in). Since the computations for $H_0(X)$ did not involve 2- simplices, we get the same result as before: $H_0(X) \cong \mathbb{R}$.

When it comes to $H_1(X)$, we still have $dim(ker\partial_1) = dim(Z_1(X)) = 1$, but now we have to deal with $B_1(X)$. We compute the boundary of our only 2-simplex just to make sure it is not a cycle:

$$\partial_2\Big((v_0v_1v_2)\Big) = \sum_{i=0}^{2}(-1)^i(v_0v_1v_2)/(v_i) = (v_1v_2) - (v_0v_2) + (v_1v_2) \neq 0$$

This means $dim(B_1(X)) = dim(Im(\partial_2)) = 1$ and so we have:

$$dim(H_1(X)) = dim\left(\frac{Z_1(X)}{B_1(X)}\right) = dim(Z_1(X)) - dim(B_1(X)) = 1 - 1 = 0$$

$\implies H_1(X) \cong 0$

For $H_2(X)$, by the rank-nullity theorem we see that

$$dim(Z_2(X)) = dim(ker\partial_2) = dim(C_2(X)) - dim(Im(\partial_2)) = 1 - 1 = 0$$

$\implies Z_2(X) = 0$ and since there are no 3-simplices, we know that $Im(\partial_3) = B_2(X) = 0$. So we end up with

$$H_2(X) = \frac{Z_2(X)}{B_2(X)}) = \frac{\{0\}}{\{0\}} \cong 0$$

Similarly $H_k(X) \cong 0$ for all $k \geq 2$.

To summarise, we have:

$$H_i(X) = \begin{cases} \mathbb{R} & \text{if } i = 0 \\ 0 & \text{if } i \geq 1 \end{cases}$$

This again matches what we see, because now our triangle has an interior, and thus no holes.

We note that since our triangle with interior is homotopy equivalent to a point, the homology of the point and the triangle with interior are the same, which means we didn't actually have to do the computation for the triangle with interior.

Theorem 5.7. *If we can find a spanning tree in our simplicial complex X, then $H_0(X) = \mathbb{R}$.*

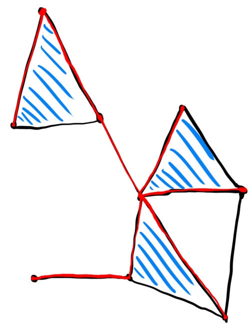

Figure 24: Example of a tree spanning a simplicial complex.

6 Persistence

Given a point cloud $\mathbb{X}$, we will want to generate a simplicial complex from this point cloud. How do we do this? From the previous ideas about generating simplicial complexes, after choosing some $\epsilon > 0$, we have a way of generating a simplicial complex, and then computing the simplcial homology and finding topolgical invariants about the complex. The problem now is, how do we choose our epsilon? Instead of choosing an epsilon, we allow epsilon to vary and look at the different simplicial complexes as epsilon varies. Instead of computing the dimensions of the homology of a simplicial complex, we compute a topological signature of a family of simplicial complexes, which is described by a diagram called a barcode. See figure 25.

These diagrams provide a topological signature of families of simplicial complexes associated with our point cloud that can help us approximate its shape.

Figure 25: Example of a Barcode. Source: [2]

6.1 Rips Example

How can we use homology to find the"shape" of point clouds? Given a point cloud $\mathbb{X}$, how do we generate a simplicial complex that will give us meaningful results?

The issue may sound expert domain specific, since for any given point cloud we could construct dozens of simplicial complexes. So which simplicial complex do we choose? If we consider the Rips complex generated from a point cloud $\mathbb{X}$ with $|\mathbb{X}| = N$

$$R(\mathbb{X})_r = \left\{ \{x_0, x_2, ...x_n\} \subseteq \mathbb{X} \;\middle|\; n \leq N \text{ and } B(x_i, r) \cap B(x_j, r) \neq \varnothing \;\; \forall i, j \in [0..n] \right\},$$

we see that we can generate a simplicial complex for each $r \in [0, +\infty)$, and so instead of choosing a fixed value for r we will instead consider the family of Rips complexes parameterized through $r \in [0, +\infty)$, denoted as

$$\{R(\mathbb{X})_r\}_{r \in [0,+\infty)}$$

or for short, $\{R(\mathbb{X})_r\}$.

We can look at some properties of $\{R(\mathbb{X})_r\}$. For instance, consider some $r_0 \in [0, +\infty)$. Then for any $r > r_0$, we have $R(\mathbb{X})_{r_0} \subseteq R(\mathbb{X})_r$, and so we can define an inclusion map

$$i : R(\mathbb{X})_{r_0} \to R(\mathbb{X})_r.$$

From there we can then consider the entire family of inclusion maps

$$i(r, r') : R(\mathbb{X})_r \to R(\mathbb{X})_{r'},$$

where we have $r, r' \in [0, +\infty)$ with $r \leq r'$. Notice that for any $r_0, r_1, r_2 \in [0, +\infty)$ with $r_0 \leq r_1 \leq r_2$, we have

$$i(r_1, r_2) \circ i(r_0, r_1) = i(r_0, r_2).$$

The families $\{R(\mathbb{X})_r\}$ and $i(r, r')$ lead us to our next definiton.

Definition 6.1 (Persistent Set)**.** Given a family of sets $\{X_r\}_{r \in A \subseteq \mathbb{R}}$ and a family of functions $\phi(r, r') : X_r \to X_{r'}$ with

$$\phi(r_1, r_2) \circ \phi(r_0, r_1) = \phi(r_0, r_2)$$

for $r_0 \leq r_1 \leq r_2$, we will call the pair $\{X_r\}_{r \in A \subseteq \mathbb{R}}$ and $\phi(r, r')$ a persistent set, and denote it as just $\{X_r\}_{r \in A}$ or $\{X_r\}$.

Examples of persistent sets can be generated from the Rips complex construction -as we did in the this section- or we can similarly generate a persistent set using the Cech complex construction.

The idea of persistence can also be applied to the witness complex construction.

6.2 Persistent Vector Spaces

We now apply the idea of persistence to vector spaces. We will end up with a family of vector spaces and a family of linear transformations. Persistent vector spaces are the foundation for defining persistent homology.

Definition 6.2 (Persistent Vector Space)**.** We define a persistent vector space over a field k as a family of vector spaces over k, denoted $\{V_r\}_{r\in[0,\infty)}$, that comes with a family of linear transformations $L_V(r,r') : V_r \to V_{r'}$ for each $r, r' \in \mathbb{R}$ with $r \leq r'$, so that for any $r, r', r'' \in \mathbb{R}$, where $r \leq r' \leq r''$, we have:

$$L(r', r'') \circ L(r, r') = L(r, r'')$$

For shorthand we will write $\{V_r\}_{r\in[0,\infty)}$ as $\{V_r\}$.

Definition 6.3 (Linear Transformation of a Persistent Vector Space)**.** For two persistent vector spaces $\{V_r\}$ and $\{W_r\}$, both over the same field k, with families of linear transformations $L_V(r,r')$ and $L_W(r,r')$ respectively, we can define a persistent linear transformation $f : \{V_r\} \to \{W_r\}$ as the family of linear transformations of the form $f_r : V_r \to W_r$, with $r \in [0,+\infty)$. Also, for f to be a persistent linear transformation, we must have that for each $r, r' \in [0,\infty)$ with $r \leq r'$, the equality

$$f_{r'} \circ L_V(r,r') = L_W(r,r') \circ f_r$$

holds.

Definition 6.4 (Definition of Persistent Subspace)**.** For a persistent vector space $\{V_r\}$ we can define a persistent subspace of $\{V_r\}$, denoted $\{U_r\}$, as the persistent vector space where for each $r, r' \in [0,+\infty)$, with $r \leq r'$, we have that U_r is a subspace of V_r and we have $L_V(r',r)(U_r) \subseteq U_{r'}$.

Definition 6.5 (Persistent Quotient Space)**.** Consider a persistent vector space $\{V_r\}$ and a persistent subspace $\{U_r\}$. Then we define $\dfrac{\{V_r\}}{\{U_r\}}$ as the persistent quotient space $\left\{\dfrac{V_r}{U_r}\right\}_{r\in[0,+\infty)}$ with the family $L_{V/U}(r,r') : \dfrac{V_r}{U_r} \to \dfrac{V_{r'}}{U_{r'}}$ defined by $L_{V/U}(r,r')([v]) = [L_V(r,r')(v)]$

Definition 6.6 (Finitely Generated Free Vector Space)**.** Given a finite set X and a field k, we can use X as a basis and generate the set

$$V_k(X) = \left\{ \sum_{x \in X} a_x x \,\middle|\, a_x \in k \right\}$$

where the operations of vector addition and scalar multiplication meet the conditions for $V_k(X)$ to be a vector space. We will call $V_k(X)$ a finitely generated free vector space on X over the field k, or just a finitely generated vector space.

How can we take the idea of persistence and apply it to $V_k(X)$?

Consider a function $\rho : X \to [0, \infty)$. Using the pair (X, ρ), we can define a set $X[r] \subseteq X$ where

$$X[r] = \{x \in X \mid \rho(x) \leq r\}.$$

We can define a $V_k(X,\rho)_r \subseteq V_k(X)$ as

$$V_k(X,\rho)_r = \left\{ \sum_{x \in X[r]} a_x x \,\middle|\, a_x \in k \right\}.$$

If we change notation so that $X_r = X[r]$, and $V_k(X,\rho)_r = V_k(X_r)$, we see how we can apply persistence to $V_k(X)$:

Definition 6.7 (Free Persistent Vector Space)**.** Starting with the pair (X, ρ) we generate the family $\{V_k(X,\rho)_r\} = \{V_k(X_r)\}_{r \in [0,+\infty)}$ and use any family of linear transformations $L(r,r')$ that satisfies the defintion of persistent vector spaces. Thus we are left with the persistent vector space $\{V_k(X_r)\}$ generated from(X, ρ).

Definition 6.8 (Finitely Generated Free Persistent Vector Space)**.** For a persistent vector space $\{V_k(X,\rho)_r\}$, if X is finite, then we will call $\{V_k(X,\rho)_r\}$ finitely generated.

Definition 6.9 (Finitely presented Free persistent vector space)**.** Given a persistent linear transformation

$$f : \{V_r\} \to \{W_r\}$$

where $\{V_r\}$ and $\{W_r\}$ are both finitely generated, we will call a persistent vector space isomorphic to the form

$$\frac{\{W_r\}}{im(f)}$$

finitely presented.

Finitely presented persistent vector spaces are very important because they can be classified up to isomophism.

First we consider a special finitely presented persistent vector space.

Definition 6.10 ($P(a,b)$)**.** For any $a \in [0,+\infty)$, and for any $b \in [0,+\infty) \cup \{+\infty\}$ with $a < b$, we can define a persistent vector space $P(a,b)$ over a field k where $P(a,b)$ is defined by

$$P(a,b)_r = \begin{cases} k & \text{if } r \in [a,b) \\ \{0\} & \text{otherwise} \end{cases}$$

with

$$L(r,r') = \begin{cases} Id_k & \text{if } r, r' \in [a,b) \\ 0_{map} & \text{otherwise} \end{cases}$$

It can be seen that $P(a,b)$ is of the form $\frac{\{W_r\}}{im(f)}$ with $f : \{V_r\} \to \{W_r\}$ by setting the generating set of $\{V_r\}$ to be a one element set $\{v\}$ with $\sigma(v) = b$ and where $\{W_r\}$ is generated by the one element set $\{w\}$ with $\rho(w) = a$. When we have our persistence parameter $r = a$, we have $P(a,b)_r = \frac{W_r}{im(f_r)} = \frac{k}{0} \cong k$. When we have our persistence parameter $r = b$,if f is not the zero map, we have $P(a,b)_r = \frac{W_r}{im(f_r)} = \frac{k}{k} \cong 0$ We see that if $b = +\infty$, then $P(a,b)_r = k$ for all $r \geq a$.

Now we have theorems from [2] which guarantee that we can classify all finitely presented persistent vector spaces.

Theorem 6.11. *Every finitely presented persistent vector space is isomorphic to a finite direct sum of the form*

$$P(a_1,b_1) \oplus P(a_2,b_2) \oplus ... \oplus P(a_n,b_n)$$

where $a_i \in [0,+\infty)$*,* $b_i \in [0,+\infty]$*, and* $a_i < b_i$ *for all* i*.*

Theorem 6.12. *Given a finitely presented persistent vector space $\{V_r\}$ over the field k, suppose we have two decompositions where*

$$\{V_r\} \cong \bigoplus_{i \in I} P(a_i, b_i) \text{ and } \{V_r\} \cong \bigoplus_{j \in J} P(c_j, d_j)$$

where I and J are fintie indexing sets. Then $|I| = |J|$ and the set of pairs (a_i, b_i) with multiplicities is the same as the set of pairs (c_j, d_j) with multiplicities.

So we have for each finitely presented persistent vector space exactly one direct sum determined by a sum of pairs (a_i, b_i) with multiplicities. From the decomposition we will see how we can visually represent finitely presented persistent vector spaces with intervals.

7 Persistent Homology

We can now compute persistent homology. Given a filtered simplicial complex with a family of inclusions, we can generate a persistent vector space of $\{C_k(X_r)\}$ for each k, with a family of linear transformations generated from the inclusions of $\{X_r\}$, and then we can consider the persistent sub vector spaces $\{Z_k(X_r)\}$ and $\{B_k(X_r)\}$. and take the persistent quotient and define the k-persistent homology.

Definition 7.1. Consider the filtered simplicial complex $\{X_r\}$ with the the family of inclusion maps $\{i_{r,r'}\}$. We can then consider the persistent vector space $\{C_k(X_r)\}$ with the family of linear transformations $\{i^C_{r,r'}\}$ where for each r, r' with $r < r'$, $i^C_{r,r'} : C_k(X_r) \to C_k(X_{r'})$ is defined for any $x = \sum_{\sigma \in X_r} a_\sigma \sigma$ by

$$i^C_{r,r'}(x) = i^C_{r,r'}(\sum_{\sigma \in X_r} a_\sigma \sigma) = \sum_{\sigma \in X_r} a_\sigma i_{r,r'}(\sigma)$$

We can then consider the persistent vector space $\{Z_k(X_r)\}$ with the family of linear transformations $\{i^Z_{r,r'}\}$ which is just $i^C_{r,r'}$ restricted to cycles in $C_k(X_r)$.

Finally we can define the kth persistent homology as the the persistent vector space

$$\{H_k(X_r)\} = \frac{\{Z_k(X_r)\}}{\{B_k(X_r)\}}$$

with the family of linear transformations $i^H_{r,r'} : H_k(X_r) \to H_k(X_{r'})$ defined by

$$i^H_{r,r'}([z]_r) = [i^Z_{r,r'}(z)]_{r'}$$

We can see the quotient previously mentioned is actually a finitely presented persistent vector space. This means our classification theorems apply to it. We now see how:

First we have our finite filtered simplicial complex. Let's call it $\{X_r\}$. The linear transformations will be the family of inclusions $i(r, r') : X_r \to X_{r'}$. From here we should define our boundary operator. Consider the persistent linear transformation $\partial : \{C_{i+1}(X_r)\} \to \{C_i(X_r)\}$. First we notice that since our filtered simplicial complex has a finite base set X, we know there is some N such that for all $r \geq N$, we have $X_r = X_N = X$.

We will say our filtered simplcial complex stabilizes to X.

Now consider $\partial_N : C_{i+1}(X) \to C_i(X)$ and $\partial_r : C_{i+1}(X_r) \to C_i(X_r)$. Let ∂' equal ∂_N restricted to X_r. We can do this since we have $X_r \subseteq X$ for all r. Then we notice

$$\partial' = \partial_r.$$

From[2], this allows us to use a matrix that represents ∂_N. This matrix determines a finitely presented vector space isomorphic to a persistent vector space of the form

$$P(a_1, b_1) \oplus P(a_2, b_2) \oplus ... \oplus P(a_n, b_n),$$

and from this decomposition we can determine a barcode to for the ith persistent homology.

7.1 Barcodes

We can represent our persistent homology using intervals. See right side of figure 26. If we have $P(a, b)$ as a term in our direct sum, we can draw a line from a to b in a diagram called a barcode.

When we compute the barcode, one rule that we use is that long intervals correspond to the structure of the data, while short ones don't.

In figure 26, the persistence parameter varies along the horizontal axis of the barcode diagram. The left endpoint of each bar represents a new feature appearing, and each right endpoint of each bar represents a feature disappearing.

For example, if we compute the first persistent homology of a statistical circle, we should have one long bar corresponding to the the hole of the circle. See figure 26.

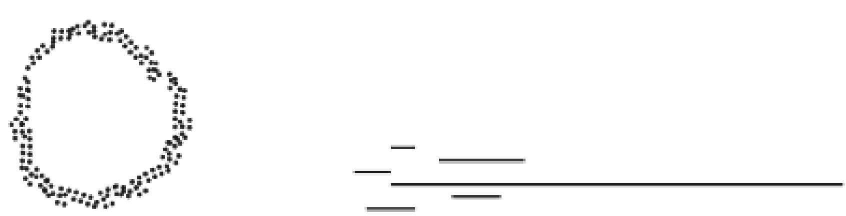

Figure 26: Left: Statistical Circle Right: Barcode generated from Statistical Circle. Source: [2] Topological Data Analysis

7.2 The Space of Barcodes

Given a point cloud we can compute the persistent homology of a filtered simplicial complex. We will focus on a result that applies to the Rips Complex construction. We want to see how small changes in data affect our barcodes. This means we need to be able to define precislely what it means for to barcodes to differ. We do this by considereing the set of all barcodes, $\mathbb{B}$, and then we define a metric d_B, called the bottleneck distance, on $\mathbb{B}$.

We can define $\mathbb{B}$,the set of all possible barcodes as $(\mathbb{R}_+ \cup \infty)^2$ with the condition that for any point $(x_0, x_1) \in \mathbb{B}$ we have $x_0 \leq x_1$. A barcode is a subset of $\mathbb{B}$.

Definition 7.2 (Bottleneck Distance)**.** Given any $X, Y \subseteq \mathbb{B}$, we can define the bottleneck distance d_B between X and Y:

$$d_B(X, Y) = inf_\eta sup_x ||x - \eta(x)||_\infty$$

where η runs through all possible bijections between X and Y, with the detail that if there are not enough points in X or Y to have bijections, we can use the closest point on the main diagonal $\{(p_1, p_2) \in \mathbb{R}^2 \mid p_1 = p_2\}$ to create bijections whenever there is a point in X or Y that doesn't have a point to be mapped η to or have a point that receives nothing through η, respectively.

This is one way to define the bottlneck distance. We can define another way. First we have to define some other ideas.

Definition 7.3 ($\Delta(I, J)$)**.** Given two intervals $I = [x_1, y_1]$ and $J = [x_2, y_2]$ from $\mathbb{R}$, we define $\Delta(I, J)$ with

$$\Delta(I, J) = d_\infty(I, J)$$

where we are treating I and J as points in $\mathbb{R}^2$ in the right side of the equation. Remember that for any two points (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$, we have

$$d_\infty((x_1, y_1), (x_2, y_2)) = max\{|x_2 - x_1|, |y_2 - y_1|\}$$

Definition 7.4 ($\lambda(I)$)**.** For any interval $I = [x, y]$, we define $\lambda(I)$ with

$$\lambda(I) = \frac{y - x}{2}$$

Definition 7.5 ($P(\theta)$)**.** Given to families of intervals $\mathbb{I} = \{I_\alpha\}_{\alpha \in A}$ and $\mathbb{J} = \{J_\beta\}_{\beta \in B}$, and any bijection $\theta : A' \to B'$, for some $A' \subseteq A$ and some $B' \subseteq B$, we define $P(\theta)$ with

$$P(\theta) = max\Big(max_{a \in A'}\big(\Delta(I_a, J_{\theta(a)})\big), max_{a \in A - A'}\big(\lambda(I_a)\big), max_{b \in B - B'}\big(\lambda(J_b)\big)\Big)$$

Definition 7.6 (Alternative definition of bottleneck distance)**.** We now define the bottleneck distance $d_\infty : \mathbb{B} \times \mathbb{B} \to \mathbb{R}$ with

$$d_\infty(\mathbb{I}, \mathbb{J}) = min_\theta P(\theta)$$

Now that we have a metric on $\mathbb{B}$, let Z be any metric space. We can define a metric on the set of all compact subsets of Z.

Definition 7.7 (Hausdorff Distance)**.** For any metric space Z, let X, Y be any compact subsets of Z. Then we define the hausdorff distance d_H with

$$d_H(X, Y) = max\{max_{x \in X} min_{y \in Y} d_Z(x, y), max_{y \in Y} min_{x \in X} d_z(x, y)\}$$

Definition 7.8 (Gromov-Hausdorff Distance)**.** For any two compact metric spaces X and Y, we define the Gromov-Hausdorff distance between X and Y as

$$inf_{f,g,Z} d_H\big(f(X), g(Y)\big),$$

where f runs through all possible isometries from X to Z, where g runs through all possible isometries from Y to Z, and where Z runs through all possible metrics spaces.

If we consider the barcodes of two finite metric spaces (point clouds) X and Y, and the barcodes $\mathbb{I}$ and $\mathbb{J}$ generated from the filtered Rips complexes $R(X)$ and $R(Y)$ we get the following:

Theorem 7.9 (Lower Bound on d_{GH})**.** *Let $\mathbb{I}$ and $\mathbb{J}$ be barcodes generated from the filtered Rips complexes $R(X)$ and $R(Y)$, respectively. Then we have the following inequality:*

$$d_B(\mathbb{I}, \mathbb{J}) \leq d_{GH}(X, Y)$$

This gives us information on how changes in point clouds affect changes in our barcodes.

7.3 Ripser

The program Ripser [4] allows us to compute the barcodes of Rips complexes associated with point clouds. The point clouds can be downloaded from [7].

Example 7.10 (Sphere)**.** *Given a point cloud $\mathbb{X}$ sampled from S^2, Ripser computes the barcode of the persistent homology associated to the Rips complex generated on $\mathbb{X}$.*

In figure 27, we see one long interval and many short intervals. The barcode shows that for small values of r, first there are many connected components, but then they join together as the parameter r increases. The long interval coincides with the visual idea that a sphere has only one connected component.

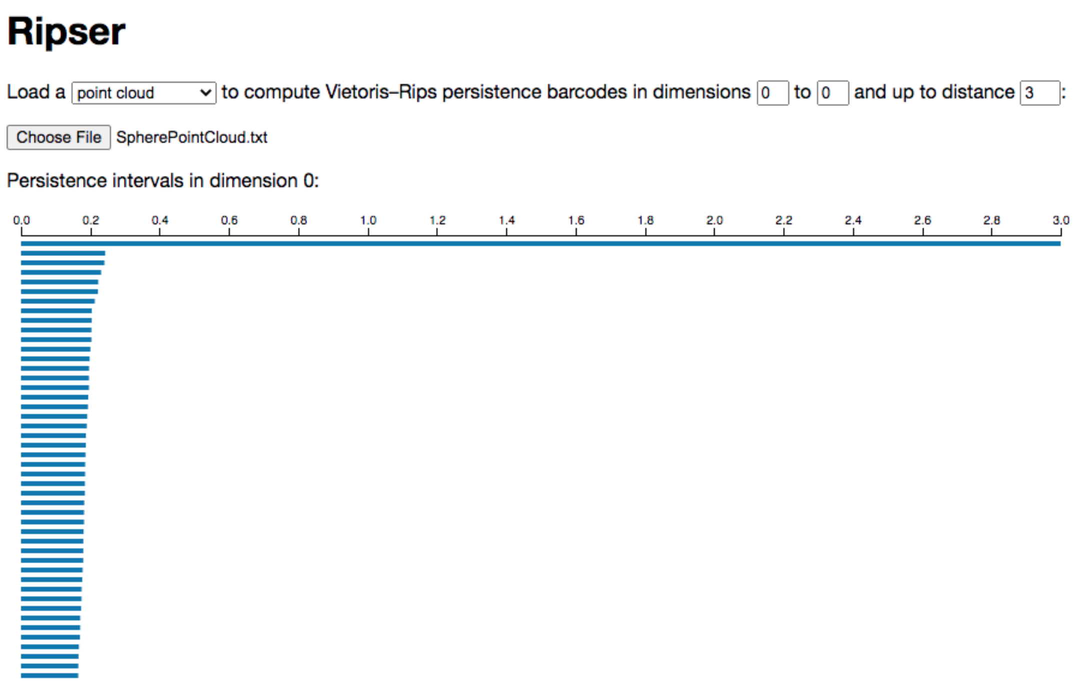

Figure 27: 0th persistent homology of a point cloud sampled from a sphere.

In figure 28, we see that there are no long intervals. This matches the idea that a sphere has no holes any holes that form as r increases quickly go away.

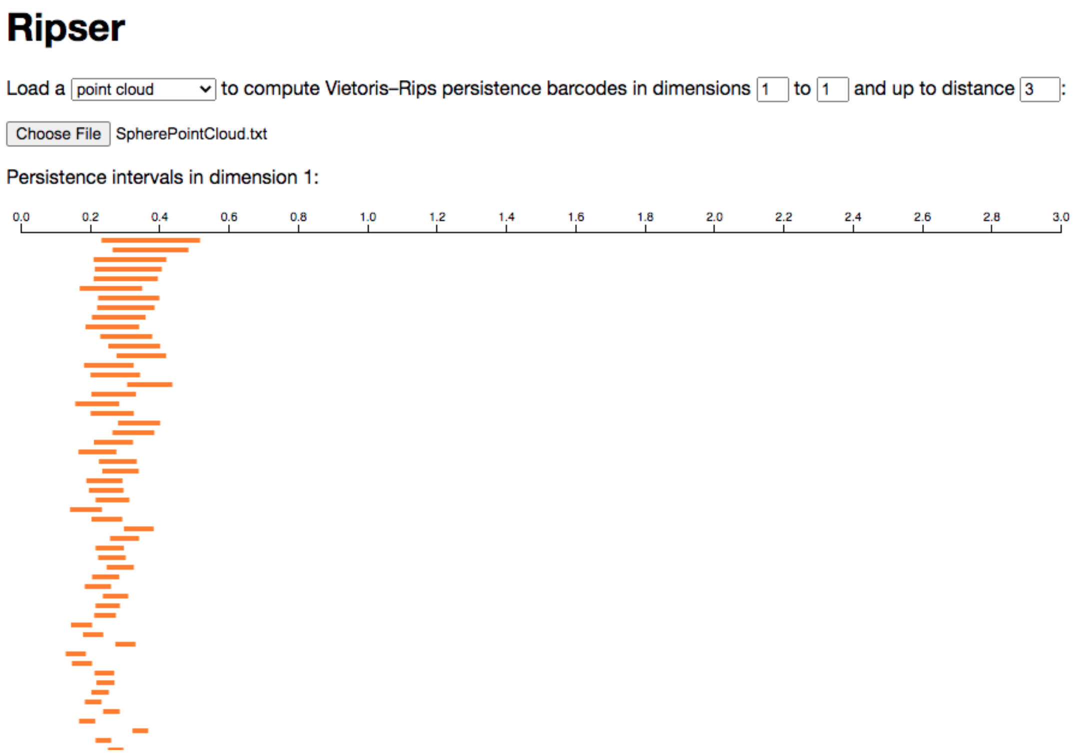

Figure 28: 1st persistent homology of a point cloud sampled from a sphere.

In figure 29, we zoomed in on the barcode, but we see that there is only one one long interval. This corresponds with the idea that a sphere is hollow and has a void inside of it. As r gets larger, we would expect this void to disappear, and the barcode corresponds with this belief.

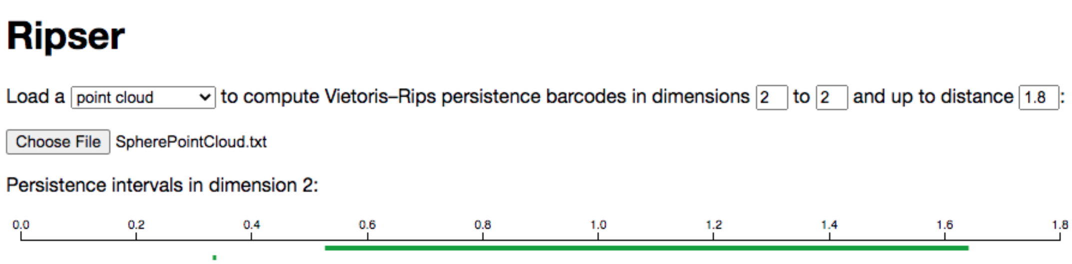

Figure 29: 2nd persistent homology of a point cloud sampled from a sphere.

Example 7.11 (Torus)**.** *Given a point cloud $\mathbb{X}$ sampled from $S^1 \times S^1$, Ripser computes the barcode of the persistent homology associated to the Rips complex generated on $\mathbb{X}$.*

We see in figure 30 one long interval, which matches the idea that a torus should have one connected component.

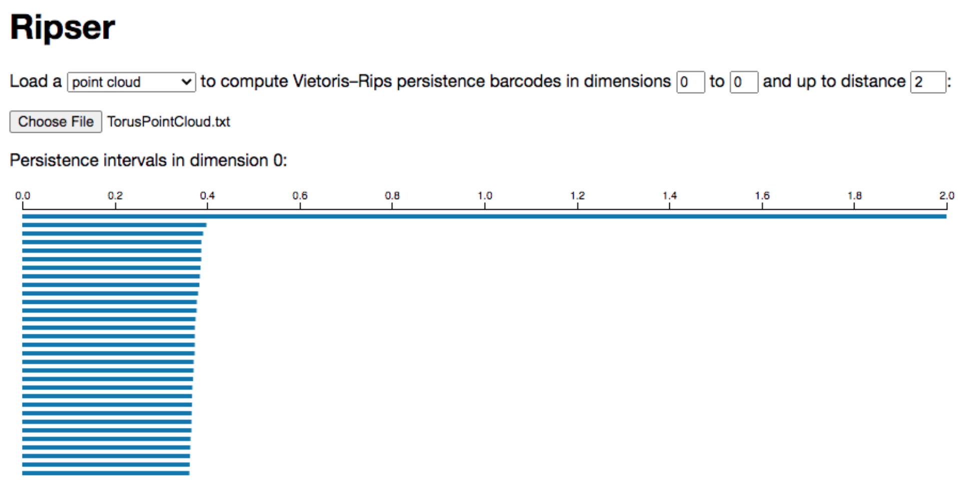

Figure 30: 0th persistent homology of a point cloud sampled from a Torus.

Next, in figure 31, we see two long intervals which matches the idea that a torus has two holes.

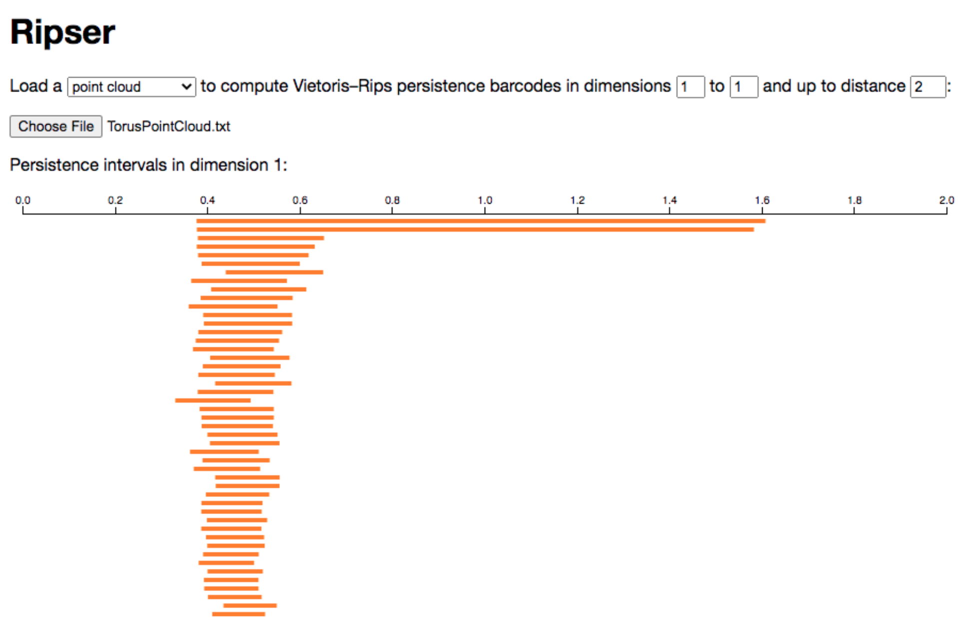

Figure 31: 1st persistent homology of a point cloud sampled from a Torus.

In figure 32, we see one long interval which matches the idea that a torus has one void.

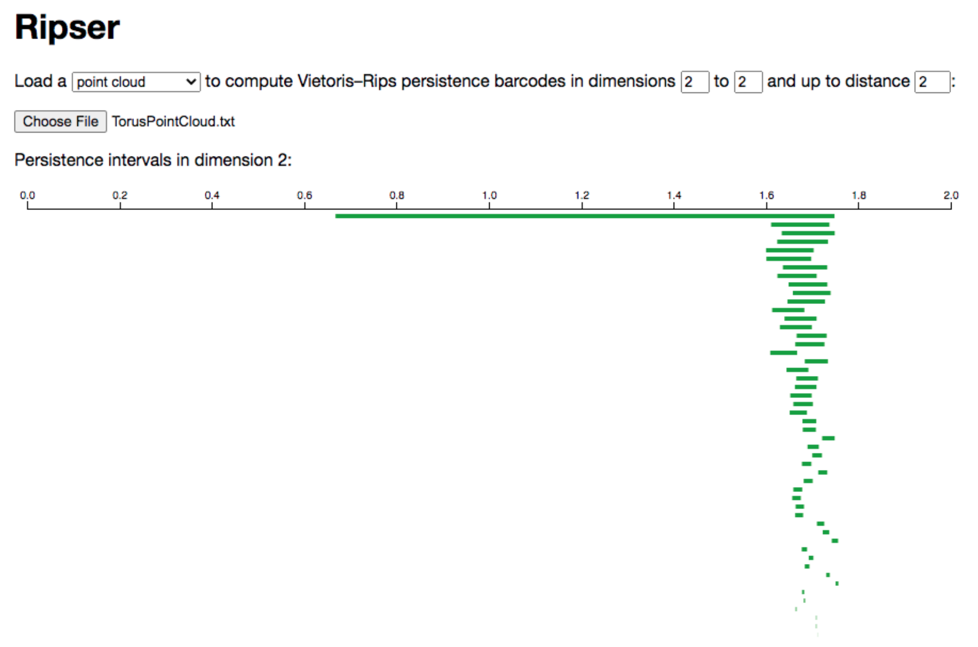

Figure 32: 2nd persistent homology of a point cloud sampled from a Torus.

8 Image Application

We now look at an application of our ideas. When we take a picture with a digital camera, our picture is actually made of of a finite collection of pixels. We will call a photo taken by a digital camera a natural image. Suppose we have a black and white natural image made of of p pixels, where each pixel can take on a value from a finite range of values. Each pixel will have a value called a grey-scale value. We can apply an ordering to the pixels and then treat it as a vector lying in $\mathbb{R}^p$. This means a digital photograph is represented by a single point in $\mathbb{R}^p$. We now note that the set of natural images which lie in $\mathbb{R}^p$ is a proper subset of all possible pixel arrangements. This is because if we think of static on a T.V. screen, we are not likely to find a digital photograph that has the same pixel arrangement.

In particular, our questions are whether or not our proper subset of $\mathbb{R}^p$, the collection of natural images, has any shape, can be modeled by some manifold, and if yes, what is this manifold? This question is addressed by [2] and [3] in the following way.

Instead of studying images in $\mathbb{R}^p$, they take 5000 3 by 3 patches from each image in the collection of 4167 images and treat each patch as a vector in $\mathbb{R}^9$, that is, as a point in a point cloud. Next they use a contrast measuring function on each each patch (a vector in $\mathbb{R}^9$), and keep the patches with contrast in the top 20% of measured contrasts. They then set constraints on the brightness and contrast of the image patches and apply a transformation, which results in point cloud which lies near S^7, in the sense that all points on S^7 lie near a point in the point cloud. Then a coordinate change is applied to the point cloud, and it concentrates into an annulus.

The co-density function δ_k is defined with $\delta_k(x) = d(x, \nu_k(x))$, where $\nu_k(x)$ is the kth nearest point to x. δ_k is inversely related to the density of the point cloud. The larger the value k the smoother the co-density function becomes, since large k means that we are checking distances to further away points, and thus ignoring the less prominent clusters that may be in our point cloud. Small k will detect these less prominent clusters.

After choosing a value for k, they use δ_k to select high density subsets $\mathcal{M}_k$ of of the point cloud, and then they compute the persistent homology of $\mathcal{M}_k$.

First they let $k = 300$. Then a witness complex $W(\mathcal{M}_{300})$ is constructed and the 1st persistent homology is computed, resulting in the barcode shown in figure 33.

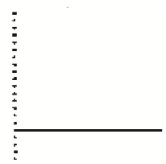

Figure 33: 1st persistent homology barcode with k=300. Source:[3]

In the first case, where we have $k = 300$, We have one long interval and many short intervals. We see how the image patches lie in figure 34.

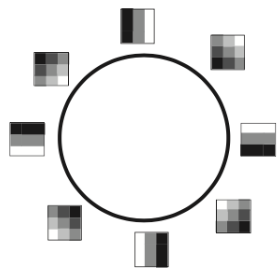

Figure 34: Known as the Primary Circle. Source:[3]

Next they let $k = 15$. We note this is smaller than the first choice of k. They then construct $W(\mathcal{M}_{15})$ and compute the 1st persistent homology barcode. See figure 35.

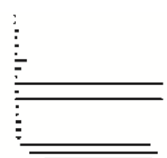

Figure 35: 1st persistent homology barcode with k=15. Source:[3]

This barcode seems to tell us there are 5 holes in our point cloud. From [2] and [3], it turns out to be modeled by following picture, figure 37, known as the 3 circle model.

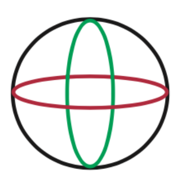

Figure 36: 3 circle model. Source:[3]

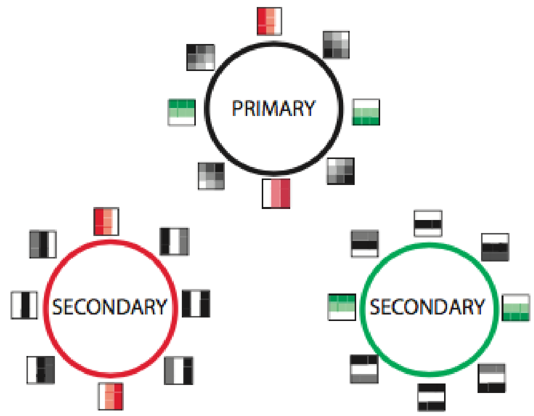

Figure 37: Types of image patches that lie on the 3 circle model. Source:[3]

We now want to see if the the 3 circle model can be embedded into a 2-manifold in a meaningful way. In [2] and [3] it is shown that there is a justifiable embedding into a Klein Bottle. See figure 38.

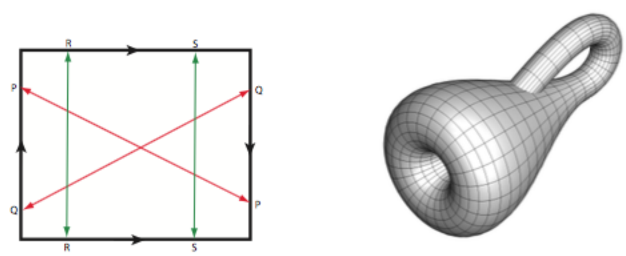

Figure 38: 3-dimensional picture of Klein Bottle. Source: [3]

After justified alterations to the point cloud data, the persistent homology is computed and results in the following barcode. See figure 39.

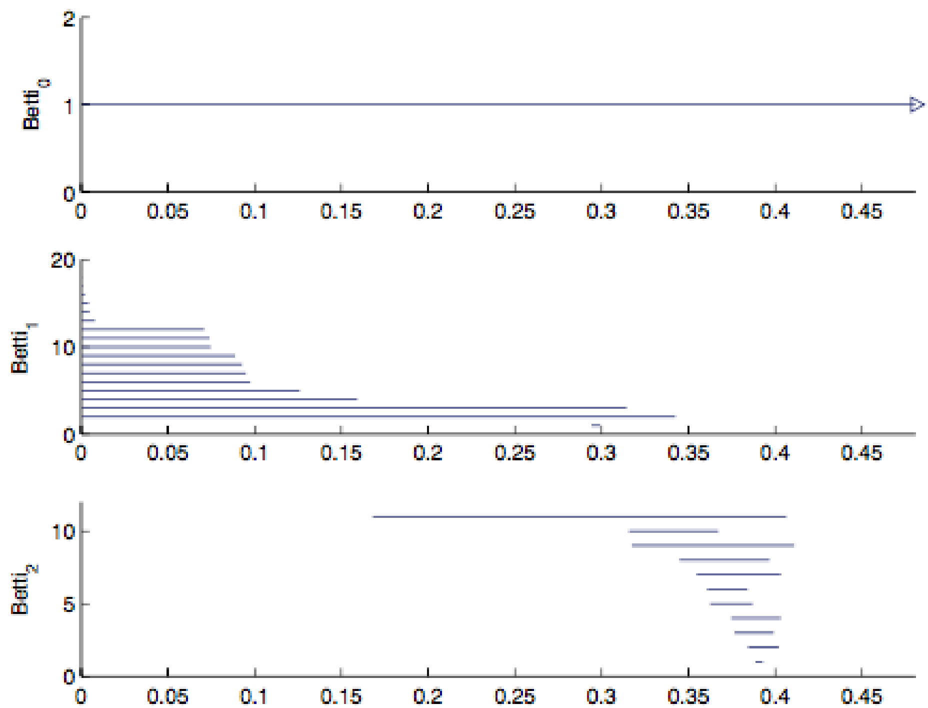

Figure 39: Persistent Homology computed using coefficients from the field $\mathbb{Z}_3$. Source:[3]

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