INTRODUCING 3-PATH DOMINATION IN GRAPHS

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Abstract. The *dominating set* of a graph G is a set of vertices D such that for every $v \in V(G)$ either $v \in D$ or v is adjacent to a vertex in D. The *domination number*, denoted $\gamma(G)$, is the minimum number of vertices in a dominating set. In 1998, Haynes and Slater introduced paired-domination. Building on paired-domination, we introduce 3-path domination. We define a *3-path dominating set* of G to be $D = \{Q_1, Q_2, \dots, Q_k | Q_i \text{ is a 3-path}\}$ such that the vertex set $V(D) = V(Q_1) \cup V(Q_2) \cup \cdots \cup V(Q_k)$ is a dominating set. We define the *3-path domination number*, denoted by $\gamma_{P_3}(G)$, to be the minimum number of 3-paths needed to dominate G. We show that the 3-path domination problem is NP-complete. We also prove bounds on $\gamma_{P_3}(G)$ and explore particular families of graphs such as Harary graphs, Hamiltonian graphs, and subclasses of trees. We leave the reader with a conjecture stating $\gamma_{P_3}(G) \leq \frac{n}{3}$.

1. Introduction

In this paper, assume any graph $G = (V, E)$ is finite and simple with vertex set V and edge set E. A *dominating set* $D \subseteq V$ of G is a set such that every vertex $v \in V$ is either in D or adjacent to a vertex in D. The *domination number* of G, denoted $\gamma(G)$, is defined as the minimum cardinality of a dominating set of G. One can think of vertices as rooms and edges as hallways. The vertices of D are rooms where guards are stationed with the ability to monitor adjacent rooms. Many variations on domination have been studied. See Haynes et al. [3] for an introduction to many of these areas. In 1998, Haynes and Slater introduced

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paired-domination [5] in which the induced subgraph on a dominating set of vertices contains a perfect matching. In this instance, we say that every guard has another guard watching their back. The *paired-domination number*, denoted $\gamma_{pr}(G)$ (originally denoted as $\gamma_p(G)$, but later changed), is the minimum cardinality of a paired-dominating set of G.

We introduce a natural extension of paired-domination, namely 3-path domination. We say Q_i is a *3-path* if it is a path on some 3 vertices $\{a, b, c\} \in V(G)$ with edges $\{ab, bc\} \in E(G)$. We define a *3-path dominating set* of G to be $D = \{Q_1, Q_2, \ldots, Q_k\}$ such that the vertex set $V(D) = V(Q_1) \cup V(Q_2) \cup \cdots \cup V(Q_k)$ is a dominating set. Here we continue our anology by allowing stationed guards to walk between 3 rooms and look down hallways extending from those 3 rooms. The *3-path domination number*, denoted $\gamma_{P_3}(G)$, is the minimum cardinality of a 3-path dominating set. We choose γ_{P_3} as notation since P_n is commonly used to refer to the path graph on n vertices (that is, the graph G with $V(G) = \{v_1, v_2, v_3\}$ and $E(G) = \{v_1v_2, v_2v_3\}$.) We will use the abbreviation γ_{P_3}-set for a minimum 3-path dominating set.

In this paper we prove that determining γ_{P_3} is NP-complete and then establish bounds on γ_{P_3} for all graphs, specifically

$$\gamma_{P_3}(G) \leq \left\lfloor \frac{n-3}{2} \right\rfloor.$$

Tighter bounds and formulas for γ_{P_3} for specific families of graphs such as caterpillars, Harary graphs, banana trees, paths, and cycles are also explored. We leave off with a conjecture,

$$\gamma_{P_3} \leq \left\lceil \frac{n}{3} \right\rceil,$$

based on results from Haynes and Slater [5] and mainly intuition.

2. The 3-path Domination Problem is NP-complete

In 1998, Haynes and Slater proved that the paired domination problem was NP-complete [5]. We generalize this result to show the 3-path domination problem is also NP-complete.

Theorem 2.1. *Deciding for a given graph H and positive integer K such that $3K \leq |V(H)|$, "Is $\gamma_{P_3}(H) \leq K$?" is NP-complete.*

Proof. We will use the known NP-complete domination problem, "For a given graph G and a positive integer K, is $\gamma(G) \leq K$?" [2]. Let $V(G) = \{v_1, v_2, ..., v_n\}$. Construct graph H by letting $V(G_i) = \{v_1^i, v_2^i, ..., v_n^i\}$ for $1 \leq i \leq 6$ and letting $v_h^i v_k^i \in E(G_i)$ if and only if $v_h v_k \in E(G)$. Let H be the graph created by these six disjoint copies of G and by adding the following edges. Let $v_h^1 v_k^2$, $v_h^3 v_k^4$, and $v_h^5 v_k^6$ be in $E(H)$ if and only if either $h = k$ or $v_h v_k \in E(G)$. Add the edges $v_h^1 v_h^3$ and $v_h^3 v_h^5$ for $1 \leq h \leq n$ to H. Thus the graph H has $6n$ vertices and can be constructed from G in polynomial time.

We claim that $\gamma(G) \leq K$ if and only if $\gamma_{P_3}(G) \leq K$.

Assume $D \subset V(G)$ is a dominating set of G with $|D| \leq K$. Let R be a set of 3-paths with $R = \{\{v_h^1, v_h^3, v_h^5\} | v_h \in D\}$. Thus R is a 3-path dominating set of H with $|D| \leq K$, so $\gamma_{P_3}(H) \leq K$.

Now, assume R is a 3-path dominating set of G with $|R| \leq K$. Let $T = \bigcup_{Q_i \in R} V(Q_i)$. Since 3-paths are not necessarily disjoint and have 3 vertices each, $|T| \leq 3K$. Thus since $G_1 \cup G_2 \cong G_3 \cup G_4 \cong G_5 \cup G_6$, we can assume that $|T_{1,2}| = |T \cap (V(G_1) \cup V(G_2))| \leq K$. Let $T^* = \{v_h^2 | v_h^1 \in T_{1,2}\} \cup (T \cap V(G_2))$. Then $|T^*| \leq |T_{1,2}| \leq K$ and T^* dominates $V(G_2)$. Hence, $\gamma(G) \leq K$. □

Having shown that the 3-path domination problem is NP-complete, we find bounds on γ_{P_3} and formulas for families of graphs.

3. Bounds On the 3-path Domination Number

We find bounds based on the parameters of a graph G, namely $\gamma(G)$, $\gamma_{pr}(G)$, and the maximum degree of a vertex in G, denoted $\Delta(G)$. Note that 3-path domination requires any component of a graph to have at least three vertices.

Theorem 3.1. *For a graph G on $n \geq 3$ vertices, $\frac{\gamma(G)}{3} \leq \gamma_{P_3}(G)$.*

Proof. Let D be γ_{P_3}-set of G and $V(D)$ be the set of vertices in D. Then $|V(D)| \leq 3|D| = 3\gamma_{P_3}(G)$ since counting 3 vertices per 3-path does not account for 3-paths that share vertices, this may result in overcounting. Furthermore, $|V(D)| \geq \gamma(G)$ as $V(D)$ forms a dominating set of G. So we have

$$\gamma(G) \leq |V(D)| \leq 3\gamma_{P_3}(G),$$

and solving yields $\frac{\gamma(G)}{3} \leq \gamma_{P_3}(G)$. □

For the next lower bound, we generalize an argument from Haynes and Slater involving $\Delta(G)$. [5]

Theorem 3.2. *For a connected graph G on $n \geq 3$ vertices, $\frac{n}{3\Delta(G)} \leq \gamma_{P_3}(G)$.*

Proof. Let D be a γ_{P_3}-set of a graph G on n vertices, and let t be the number of edges in G having one vertex in $V(D)$ and the other in

$V(G) \setminus V(D)$. Since $\Delta(G) \geq \deg(v)$ for all $v \in V(D)$ and each vertex in $V(D)$ has at least one neighbor in $V(D)$,

$$\begin{aligned} t &\leq (\Delta(G)-1)|V(D)| \\ &\leq (\Delta(G)-1)3\gamma_{P_3}(G). \end{aligned}$$

In addition, $t \geq |V(G) \setminus V(D)|$ since there is at least one edge for every vertex in G that is not in $V(D)$. So,

$$\begin{aligned} t &\geq |V(G) \setminus V(D)| \\ &= n - |V(D)| \\ &\geq n - 3\gamma_{P_3}(G). \end{aligned}$$

So we have $n - 3\gamma_{P_3}(G) \leq t \leq (\Delta(G)-1)3\gamma_{P_3}(G)$, and solving yields $\frac{n}{3\Delta(G)} \leq \gamma_{P_3}(G)$. □

Having established some lower bounds on γ_{P_3}, we explore some upper bounds.

Theorem 3.3. *For a connected graph G on $n \geq 3$ vertices, $\gamma_{P_3}(G) \leq \frac{\gamma_{pr}(G)}{2}$.*

Proof. Let G be a graph with $|V(G)| \geq 3$ and D be a γ_{pr}-set of G. Since there are $\frac{\gamma_{pr}(G)}{2}$ pairs of vertices in D, we can create a 3-path using each pair and a neighbor. This forms a 3-path dominating set with cardinality $\frac{\gamma_{pr}(G)}{2}$. □

Notice that the pairs of vertices in a γ_{pr}-set are vertex disjoint (if they are not vertex disjoint, then the induced subgraph on a γ_{pr}-set would not contain a matching, let alone a perfect matching.) This is not always the case for 3-path domination, as shown in Figure 1.

For the following lemma, we note that a 3-path dominating set of a graph G is *minimal* if the removal of any one 3-path from the set results in G no longer being dominated. A *private neighbor* of a 3-path Q is a vertex that is dominated by Q and no other 3-path.

Lemma 3.4. *For a graph G with $n \geq 3$, there exists a minimal 3-path dominating set that is edge-disjoint.*

Proof. Let G be a graph and D be a minimal 3-path dominating set on G. Suppose there are two 3-paths in D that share an edge, $Q_1 = \{v_1, v_2, v_3\}$ and $Q_2 = \{v_2, v_3, v_4\}$. Consider the following two possibilities:

Referring to Figure 2, Q_1 and Q_2 each have at least one private neighbor. If Q_1 or Q_2 do not have any private neighbors, then D is not minimal as G would still be dominated if we remove one path

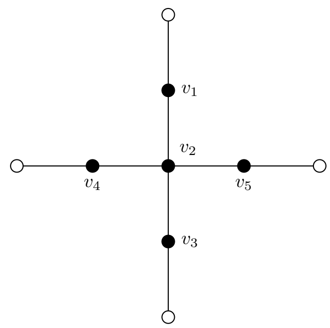

FIGURE 1. An example of a γ_{P_3}-set with $\gamma_{P_3}(G) = 2$. Notice both 3-paths must share the vertex v_2. This is independent of our set being $\{v_1v_2v_3, v_4v_2v_5\}$ or $\{v_1v_2v_5, v_4v_2v_3\}$

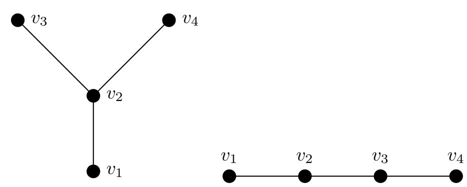

FIGURE 2. The configurations where two 3-paths, namely $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$, can share an edge.

without private neighbors. Let p_1 be a private neighbor of Q_1 and p_2 be a private neighbor of Q_2. Note, p_1 must be adjacent to v_3 on the left case and v_1 on the right case, and p_2 must be adjacent to v_4 in both cases, otherwise they would be dominated by both Q_1 and Q_2. Without loss of generality, we set $Q_1' = \{v_2, v_3, p_1\}$ in the left case, and $Q_1' = \{p1, v_2, v_3\}$ in the right case, making Q_1' and Q_2 edge-disjoint. After this change, D is not guaranteed to be minimal. If D is no longer minimal, it must be the case that Q_1' dominates what were the only private neighbors of some other 3-paths before the change. So, we can remove 3-paths that do not have any private neighbors in a way such that we obtain a new minimal 3-path dominating set D'. Repeat this process until there are no longer any edge-intersecting 3-paths. □

Corollary 3.5. *For a connected graph G on $n \geq 3$ vertices, $\gamma_{P_3}(G) \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor$.*

Proof. Let G be a graph with $|V(G)| \geq 3$ and D be a 3-path dominating set of G. Using Lemma 3.4, D can be made such that no two 3-paths in D share an edge, and so we can count one 3-path for every two unique edges. In the case that G has an odd number of edges, we count $\frac{|E(G)|-1}{2}$

pairs of edges for each 3-path. The resulting set of 3-paths would still be a dominating set, as the remaining edge, say $v_h v_k$, must be incident to one of the counted edges, and thus v_h and v_k are dominated. So we have $|D| \leq \left\lfloor \frac{|E(G)|}{2} \right\rfloor$. □

Having an upper bound in terms of the number of edges of G is useful when dealing with classes of graphs whose number of edges directly relate to the number of vertices. For example, if we consider a tree T on n vertices, $|E(T)| = n - 1$, and so $\gamma_{P_3}(T) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Using similar logic in the odd case of Corollary 3.5, we can improve this upper bound for trees. For the next corollary, we say a *leaf* of a tree is a vertex of degree 1, and the *diameter* of a graph G is the longest path between a pair of vertices in G, denoted $\operatorname{diam}(G)$.

Corollary 3.6. *For a tree T on $n \geq 3$ vertices with $\operatorname{diam}(T) \geq 4$, $\gamma_{P_3}(T) \leq \left\lfloor \frac{n-L-1}{2} \right\rfloor$ where L is the number leaves of T.*

Proof. Let T be a tree on $n \geq 3$ vertices, L be the number of leaves of T, and $\operatorname{diam}(T) \geq 4$. Using Corollary 3.5, we have $\gamma_{P_3}(T) \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Notice that we do not need to use any leaves in a 3-path, as all leaves must be adjacent to a vertex in a 3-path. So, we only need to use at most $n - L$ vertices to make our 3-path dominating set, or at most $n - L - 1$ edges. So $\gamma_{P_3}(T) \leq \left\lfloor \frac{n-L-1}{2} \right\rfloor$. □

Note that the bound does not work for $\operatorname{diam}(T) = 2$ or $\operatorname{diam}(T) = 3$. If $\operatorname{diam}(T) = 2$ then T has $n - 1$ leaves, and so we obtain $\gamma_{P_3}(T) \leq 0$ when $\gamma_{P_3}(T) = 1$. If $\operatorname{diam}(T) = 3$ then $\gamma_{P_3}(T) = 1$, however we may obtain $\gamma_{P_3}(T) \leq 0$ if T is the path graph on 4 vertices.

Consider the following observation.

Observation 3.7. *If an edge is added to a graph G to form a new graph G^*, then $\gamma_{P_3}(G^*) \leq \gamma_{P_3}(G)$.*

Intuitively, one can think of adding edges as increasing the adjacencies in a graph. The more adjacencies in a graph, the less number of 3-paths are needed to dominate the graph. Refer to Figure 3 for an example.

A *spanning tree* T of a graph G is a tree such that $V(T) = V(G)$ and $E(T) \subseteq E(G)$. We now use Observation 3.7 and spanning trees to establish a bound for general graphs.

Theorem 3.8. *For any connected graph G, $\gamma_{P_3}(G) \leq \left\lfloor \frac{n-3}{2} \right\rfloor$.*

Proof. Let G be a graph on n vertices and let T_G be its spanning tree. Notice, since $V(T_G) = V(G)$ and $E(T_G) \subseteq E(G)$, we can construct G from T_G by adding the missing edges in $E(G) \setminus E(T_G)$. By Observation

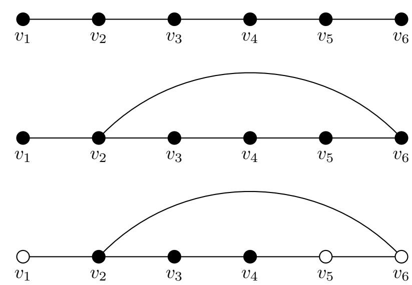

FIGURE 3. An example of how adding edges to a graph can eliminate the need for a 3-path in the resulting graph. We start with paths $\{v_1v_2v_3\}$ and $\{v_4v_5v_6\}$. When we add an edge, we end up only needing one 3-path, namely $\{v_2v_3v_4\}$.

3.7, every new graph gained by adding an edge and starting from T_G will potentially have a lower 3-path domination number. So, $\gamma_{P_3}(G) \leq \gamma_{P_3}(T_G)$, and since T_G is a tree on n vertices, $\gamma_{P_3}(T_G) \leq \left\lfloor \frac{n-L-1}{2} \right\rfloor$ where L is the number of leaves of T_G. Since $L \geq 2$ for trees with at least 2 vertices, $\gamma_{P_3}(T_G) \leq \left\lfloor \frac{n-3}{2} \right\rfloor$. □

Note that spanning trees are not necessarily unique. A graph may have several spanning trees and we can improve this bound if we are particular about our choice of a spanning tree. A *maximum leaf spanning tree* (MLST) of a graph G is a spanning tree of G that has the most leaves possible. Choosing a MLST as a spanning tree allows us to maximize L in the upper bound given in Corollary 3.6, and in turn, gain a tighter upper bound. The MLST problem is NP-complete, however Fernau et al. present a branching algorithm for finding a MLST in time $O(1.8966^n)$ [2] [1]. In addition, there are polynomial time approximation algorithms for the MLST problem [4]. As a final note, the MLST problem is analogous to the connected dominating set problem. A set D is said to be a *connected dominating set* if for every $v, u \in D$, there is a path from v to u using only vertices in D.

We present a natural conjecture for an upper bound.

Conjecture 3.9. For any connected graph G on $n \geq 3$ vertices, $\gamma_{P_3}(G) \leq \lceil \frac{n}{3} \rceil$.

We base our conjecture on intuition (as one would), as well as the following.

Theorem 3.10. *[5] If the connected graph G has $n \geq 6$ and $\delta(G) \geq 2$, then*

$$\gamma_{pr}(G) \leq \frac{2n}{3}.$$

Using Theorem 3.3 and Theorem 3.10, we can establish the following theorem.

Theorem 3.11. *If the connected graph G has $n \geq 6$ and $\delta(G) \geq 2$, then*

$$\gamma_{P_3}(G) \leq \frac{n}{3}.$$

4. Classes of Graphs and Their 3-path Domination Number

While results for all connected graphs are desirable, it is useful to restrict ourselves to families of graphs to obtain formulas and tighter upper bounds. A *path graph* denoted P_n, is a connected graph on n vertices with two vertices of degree 1, and $n-2$ vertices of degree 2. A *cycle*, denoted C_n, is a connected graph on n vertices where $\deg(v) = 2$ for every $v \in V(C_n)$.

Theorem 4.1. *For $n \geq 3$, $\gamma_{P_3}(P_n) = \gamma_{P_3}(C_n) = \lceil \frac{n}{5} \rceil$.*

Proof. Suppose we have P_n for $n \geq 3$. Say $n = 5q + k$ for nonnegative integers q and $k < 5$. If $k = 0$, we can partition P_n into q vertex-disjoint segments S_i where $|V(S_i)| = 5$ (Refer to Figure 4.) If $1 \leq k < 5$, we have q vertex-disjoint segments S_i where $|V(S_i)| = 5$, and one left over segment S_{q+1} with k vertices. We can dominate at most five vertices with a 3-path in P_n, specifically the three in the 3-path and potentially two others adjacent to the ends. In order to dominate in the best possible manner, the vertices adjacent to the ends of a 3-path must be private neighbors of the 3-path. Notice by the way we segment P_n, each 3-path will have the maximum number of private neighbors, with the exception of the additional 3-path needed to dominate the remainder segment and its adjacent segment (the 3-paths of S_q and S_{q+1} may share neighbors). So, we have dominated P_n in the best way possible.

The same logic holds for C_n, as the maximum possible number of private neighbors for a 3-path in C_n is 5. $\square$

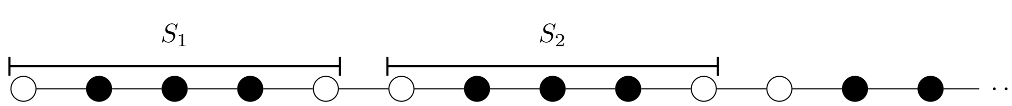

Figure 4. Path graph where black vertices are in a 3-path.

Corollary 4.2. *If G has a Hamiltonian path, then $\gamma_{P_3}(G) \leq \lceil \frac{n}{5} \rceil$.*

Proof. Suppose P_n is a Hamiltonian path of G. Since $\gamma_{P_3}(P_n) = \lceil \frac{n}{5} \rceil$ and G is obtained from P_n by adding edges, by Observation 3.7, we find that $\gamma_{P_3}(G) \leq \lceil \frac{n}{5} \rceil$. □

Introducing even the slightest complexity to a graph can hinder our ability to find a formula for $\gamma_{P_3}(G)$. One such exampleis the caterpillar tree. A *caterpillar* is a tree in which every leaf is adjacent to a central path, or stalk. See Figure 5 for an example.

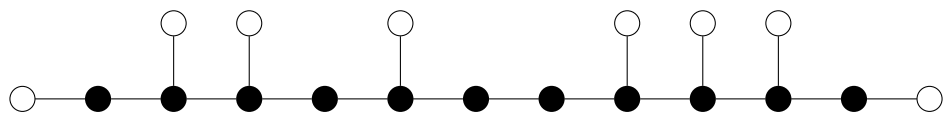

FIGURE 5. An example of a caterpillar with central stalk having 11 vertices (black). The removal of leaves yields the path graph P_{11}.

We will use the following observations in order to prove the next theorem.

Observation 4.3. *If a vertex u is adjacent to a vertex of degree 1 in $V(G)$, then u must be part of a dominating 3-path.*

Observation 4.4. *If a new vertex is connected by a single edge to a graph G to form a new graph G^*, then $\gamma_{P_3}(G) \leq \gamma_{P_3}(G^*)$*

Theorem 4.5. *Let A be a caterpillar with stalk S and let $m = |V(L)|$. Then $\lceil \frac{m+2}{5} \rceil \leq \gamma_{P_3}(A) \leq \lceil \frac{m}{3} \rceil$.*

Proof. Let A be a caterpillar with stalk S such that $|V(S)| = m$ and L leaves. Then there exists a longest path in A, P_{m+2}, such that $V(L) \subseteq P_{m+2}$. Any vertex in $V(A \setminus P_{m+2})$ must be adjacent to a vertex of P_{m+2}. Thus $\gamma_{P_3}(A) \geq \gamma_{P_3}(P_{m+2})$ by Observation 4.4. By Theorem 4.1, $\gamma_{P_3}(P_{m+2}) \geq \lceil \frac{m+L}{5} \rceil$.

Let A be a caterpillar such that every $v_i \in V(S)$ is adjacent to a leaf. Then, each of the v_i must be part of a 3-path by Observation 4.3. Furthermore, since every vertex of A is either in $V(S)$ or adjacent to a vertex in $V(S)$, $V(S)$ is a dominating set of A. So we pick our 3-paths Q_i such that that they are vertex disjoint, or $V(Q_1) \cap V(Q_2) \cap \cdots = \emptyset$ for every Q_i except for possibly two. Notice that if m is not a multiple of 3, then two of our 3-paths cannot be vertex disjoint. The least number of 3-paths we can take in this case is $\lceil \frac{m}{3} \rceil$. So the maximum number of 3-paths needed for a caterpillar is at most $\lceil \frac{m}{3} \rceil$. □

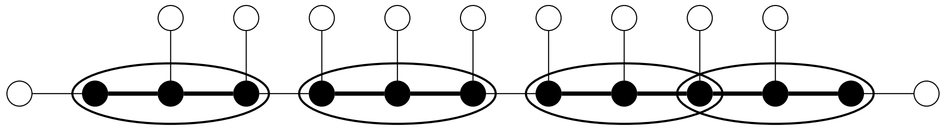

FIGURE 6. A caterpillar in which every vertex that is not a leaf is adjacent to a leaf. Black vertices are those that must be included in a 3-path, and so we partition the vertices in groups of 3 (with the possible exception of two 3-paths) to use the least number of 3-paths as possible.

A *banana tree*, denoted $B_{n,k}$, is a tree composed of n copies of a $K_{1,k}$ graph in which one leaf from each copy is joined by an edge to a vertex called the root vertex (See Figures 7 and 8 for examples).

Theorem 4.6. *The following formulas hold for* $\gamma_{P_3}(B_{n,k})$.

(1) *For* $k = 1$ *and* $n \geq 2$, $\gamma_{P_3}(B_{n,1}) = 1$.
(2) *For* $k = 2$, $\gamma_{P_3}(B_{n,2}) = \lceil \frac{n}{2} \rceil$.
(3) *For* $k \geq 3$, $\gamma_{P_3}(B_{n,k}) = n$.

Proof. (1) For $k = 1$ and $n \geq 2$, $B_{n,1}$ is a star, $K_{1,n}$. Any 3-path in a star will contain the center vertex and thus dominate all vertices. Hence, $\gamma_{P_3}(B_{n,1}) = 1$.

(2) For $k = 2$, we pick our 3-paths in the following process. By Observation 4.3, every vertex adjacent to a leaf must be in a 3-path. In the case of $B_{n,2}$, this will be the vertices adjacent to the root vertex. Since including leaves in a 3-path will not dominate anything new, we choose to include the root vertex as part of all 3-paths. Then, we make unique pairs of support vertices. If n is even, all have a pair. If n is odd, then the last 3-path will include the nth star's leaf. See Figure 7.

(3) For $k \geq 3$, by Observation 4.3, the center vertex of every copy of $K_{1,k}$ must be part of a 3-path. It is impossible to include any two centers in the same 3-path. So we obtain a 3-path for each copy of $K_{1,k}$, and we make sure that at least one 3-path contains the root vertex. So, since we have n copies, $\gamma_{P_3}(B_{n,k}) = n$. See Figure 8.

□

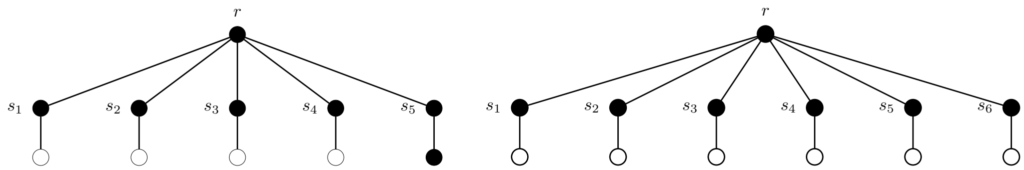

FIGURE 7. The banana trees $B_{5,2}$ and $B_{6,2}$. We pair up the support vertice s_i each with middle vertex r to form a 3-path. In the n odd case we use the leaf of the last 3-path.

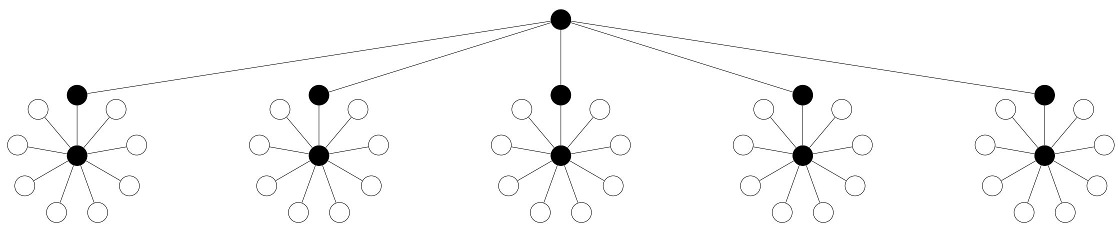

FIGURE 8. The banana tree $B_{5,10}$, where dominated vertices are white, red vertices are in a 3-path, and 3-paths are indicated by colored edges.

A *Harary graph*, denoted $H_{k,n}$, is a k-regular graph of order n with $k \leq n-1$ and $V(G) = \{v_1, v_2, \ldots, v_n\}$. If k is even, then $k = 2j$ for some j and we join v_i to $\{v_{i-j}, v_{i-j+1}, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{i-1+j}, v_{i+j}\}$. If k is odd, then $k = 2j+1$ for some j and $n = 2\ell$ for some ℓ, and we join v_i to $\{v_{i-j}, v_{i-j+1}, \ldots v_{i-1}, v_{i+1}, \ldots, v_{i-1+j}, v_{i+j}\}$ and $v_{i+\ell}$. See Figure 9 for an example.

Theorem 4.7. *For k even, we have $\gamma_{P_3}(H_{k,n}) = \lceil \frac{n}{2k+1} \rceil$.*

Proof. Suppose we have $G = H_{k,n}$ for k even. To construct a 3-path Q, we choose any vertex m to be the middle vertex of our 3-path and then choose neighbors t_1, t_2 such that there is a path of length $\frac{k}{2}$ from m to t_i (farthest on the cycle composed of $\{v_1, \ldots, v_n\}$ and $\{v_1v_2, v_2v_3, \ldots v_{n-1}v_n, v_1v_n\}$. Notice, m dominates $k+1$ vertices, and each of the t_i have an additional $\frac{k}{2}$ private neighbors so long as n is sufficiently big. So, Q dominates $k+1+\frac{k}{2}+\frac{k}{2} = 2k+1$ vertices. Note, Q dominates the largest number of vertices possible by a 3-path in G, since if we choose the t_i that are closer on the cycle to m, they would share more neighbors with m, hence dominating less vertices uniquely. In addition, all the dominated vertices lie on a single path (Figure 9). We can partition G into segments of $2k+1$ vertices and possibly one segment with less than $2k+1$ vertices if $2k+1$ does not divide n. We choose the 3-path for each of those segments, and thus we can dominate with $\lceil \frac{n}{2k+1} \rceil$ 3-paths and no fewer. □

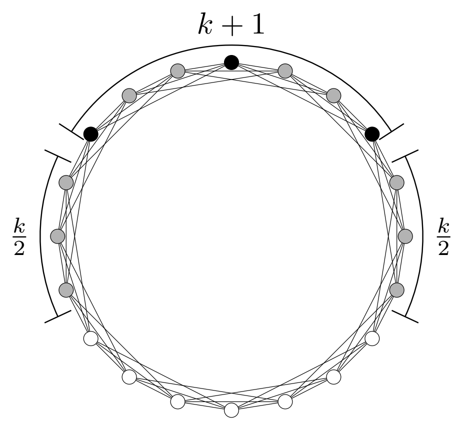

FIGURE 9. The Harary Graph $H_{6,20}$ where the black vertices are in a 3-path and the gray vertices are dominated vertices that are not in a 3-path.

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