The Magic Theorem
There are only 17 Symmetric Planar Patterns

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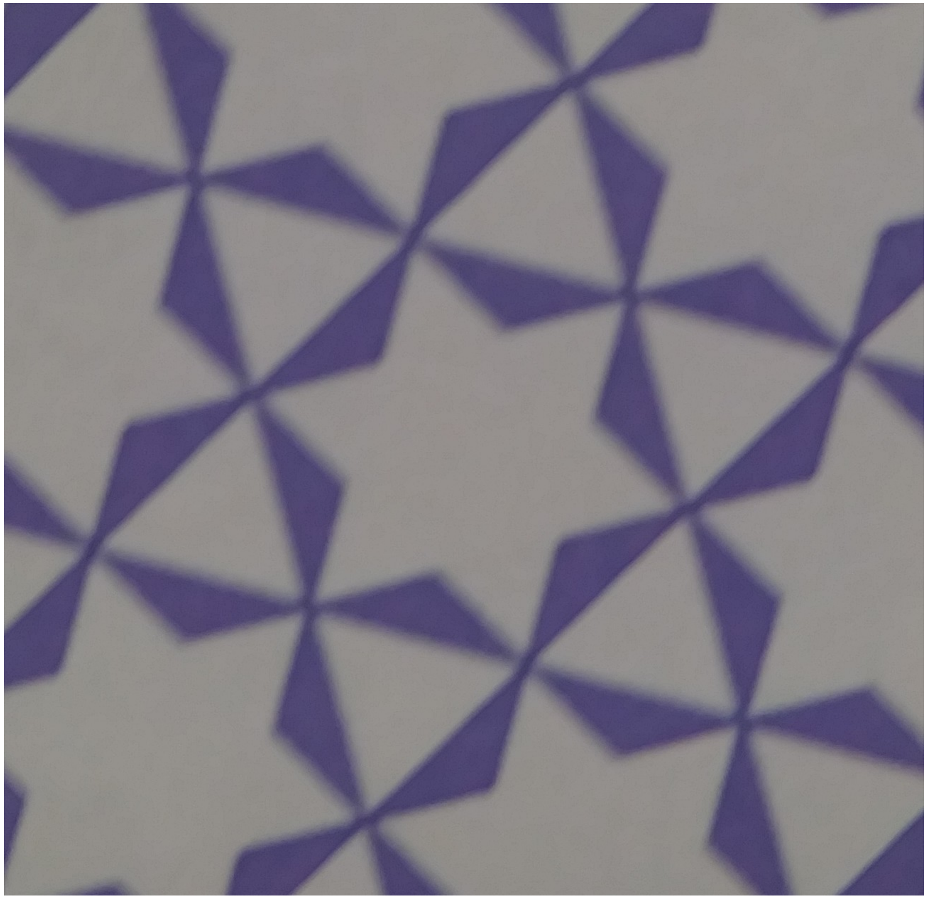
MHC – TEHA
Senior Thesis Presentation

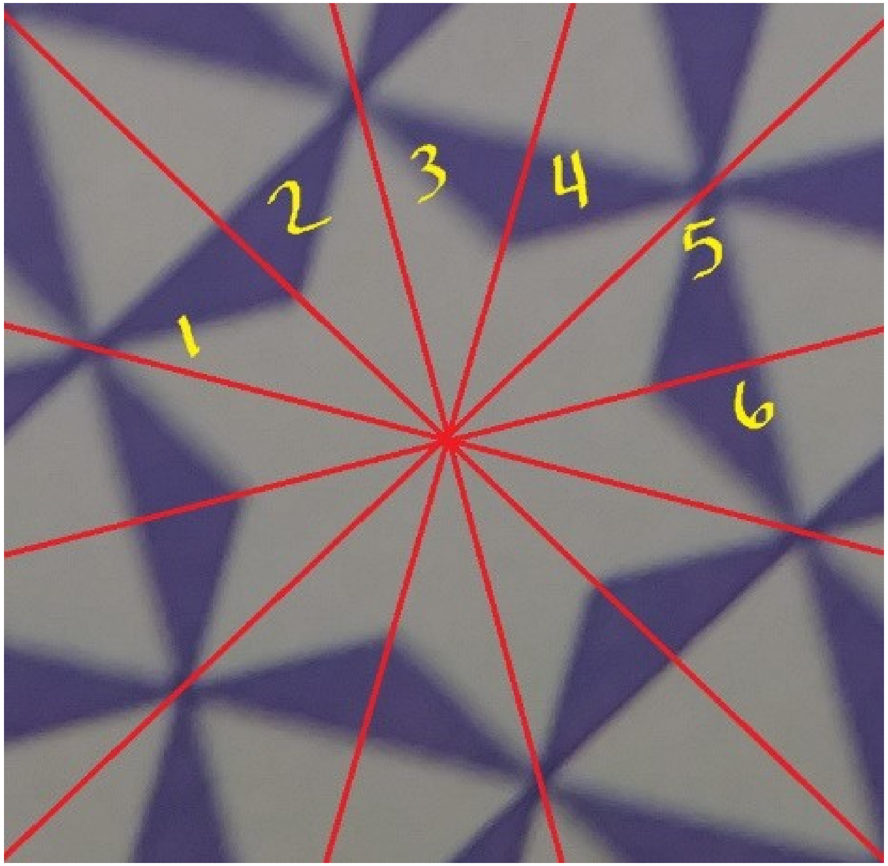
Finding the Signature of a Pattern



Observe that this pattern has reflection symmetry.

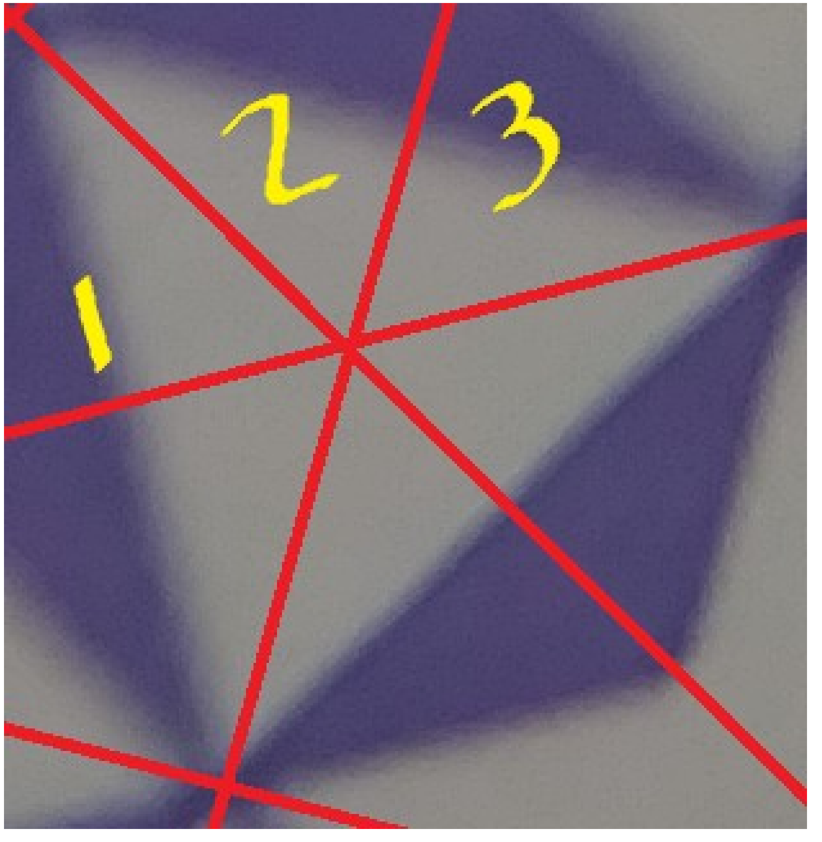
All signatures of reflection symmetry begin with *

Consider the 6-triangle Star



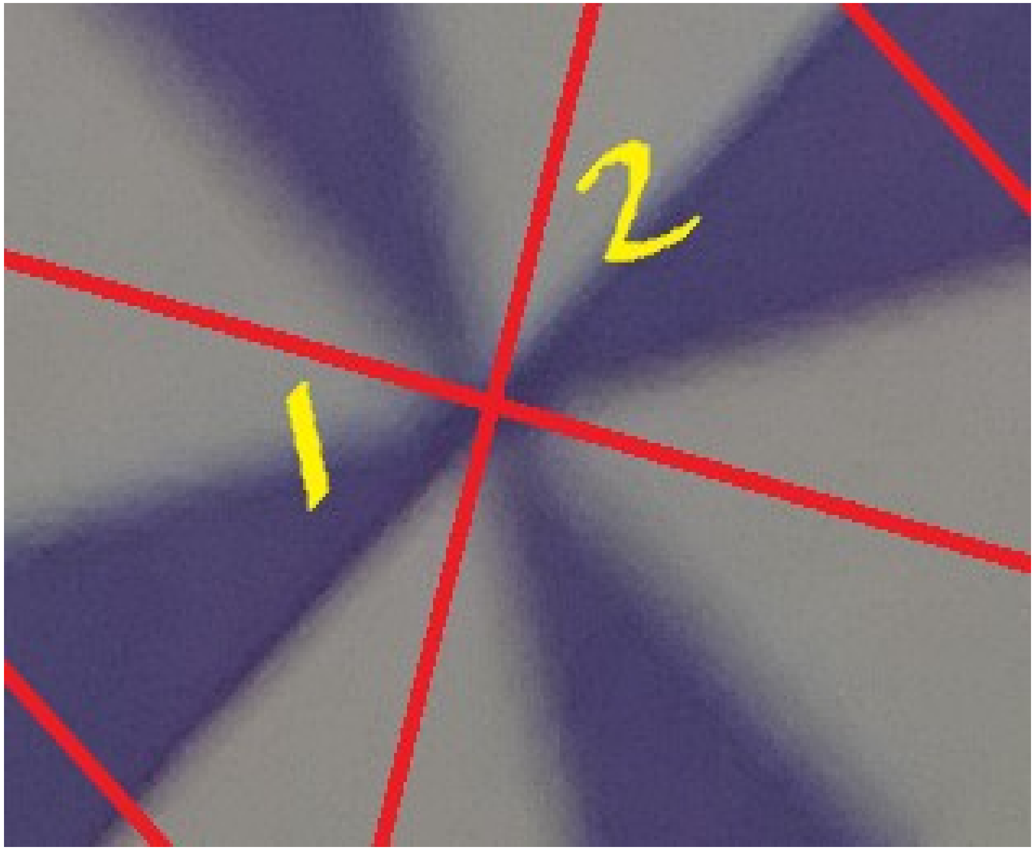
There are 6 unique lines of reflection intersecting at a point

Consider the Triangle



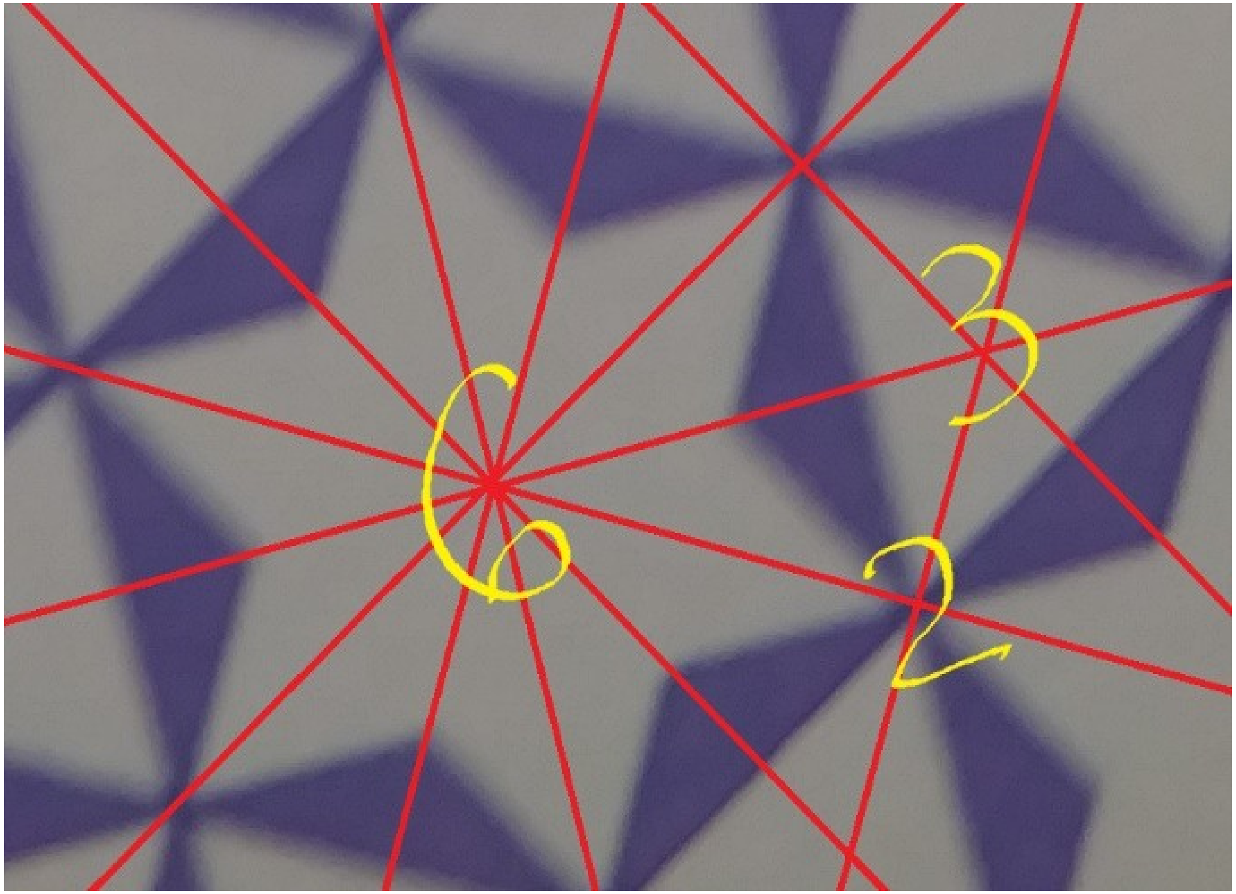
There are 3 unique lines of reflection intersecting at a point

We also find where the triangles intersect,



There are 2 unique lines of reflection intersecting at a point

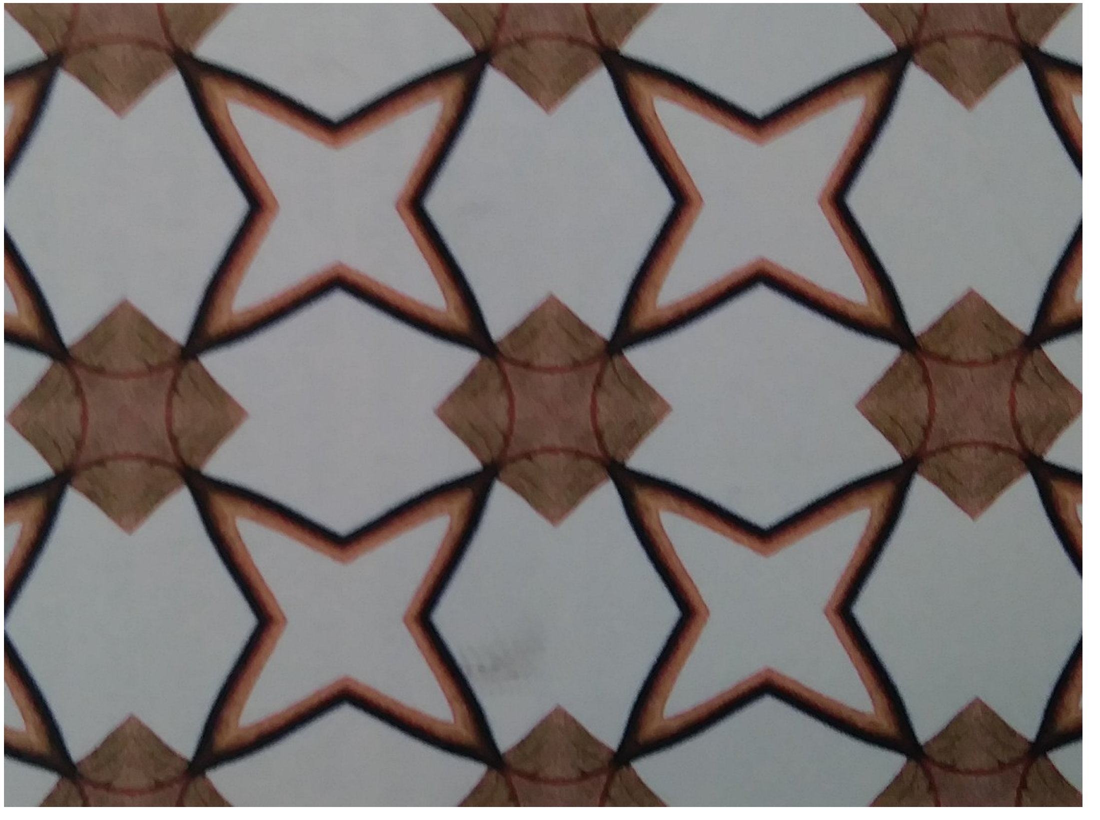
Naming the Signature



This is a reflection symmetry with one point order 6, one point order 3, and one point order 2.

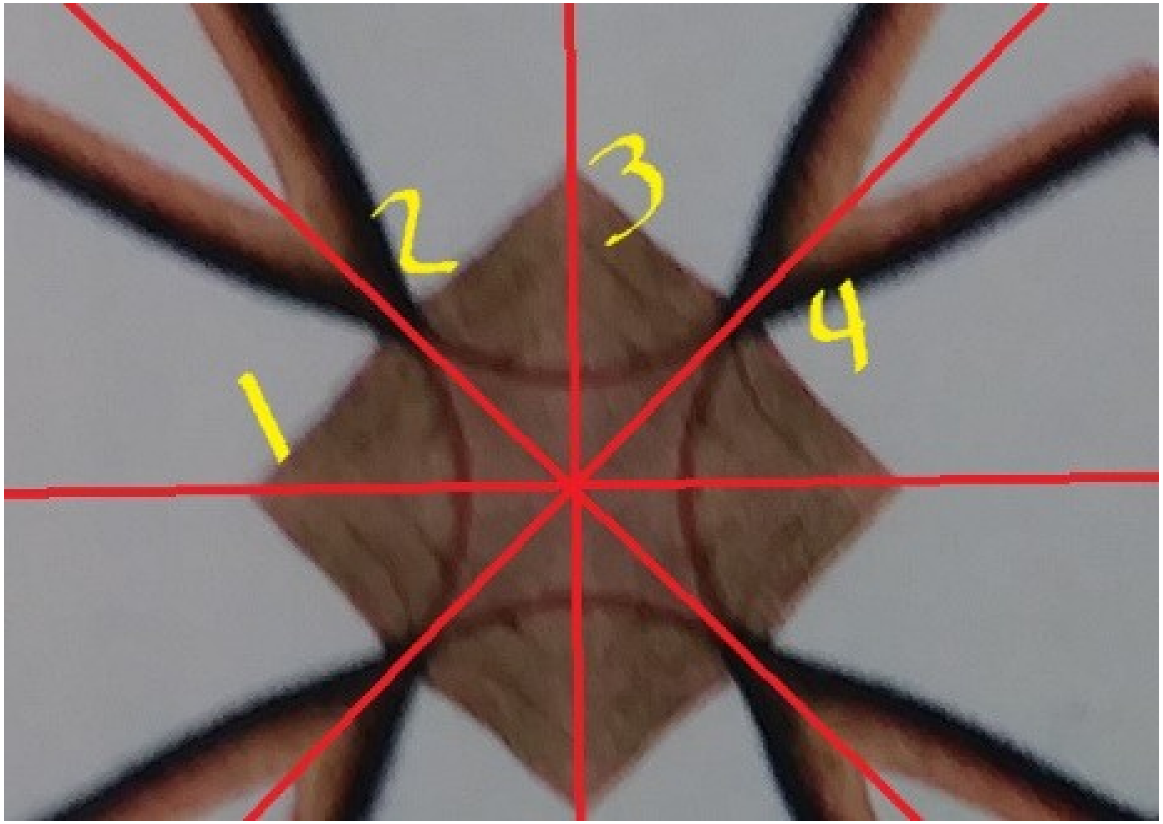
Therefore the signature is *632

What happens if there are two points with the same order of reflections?



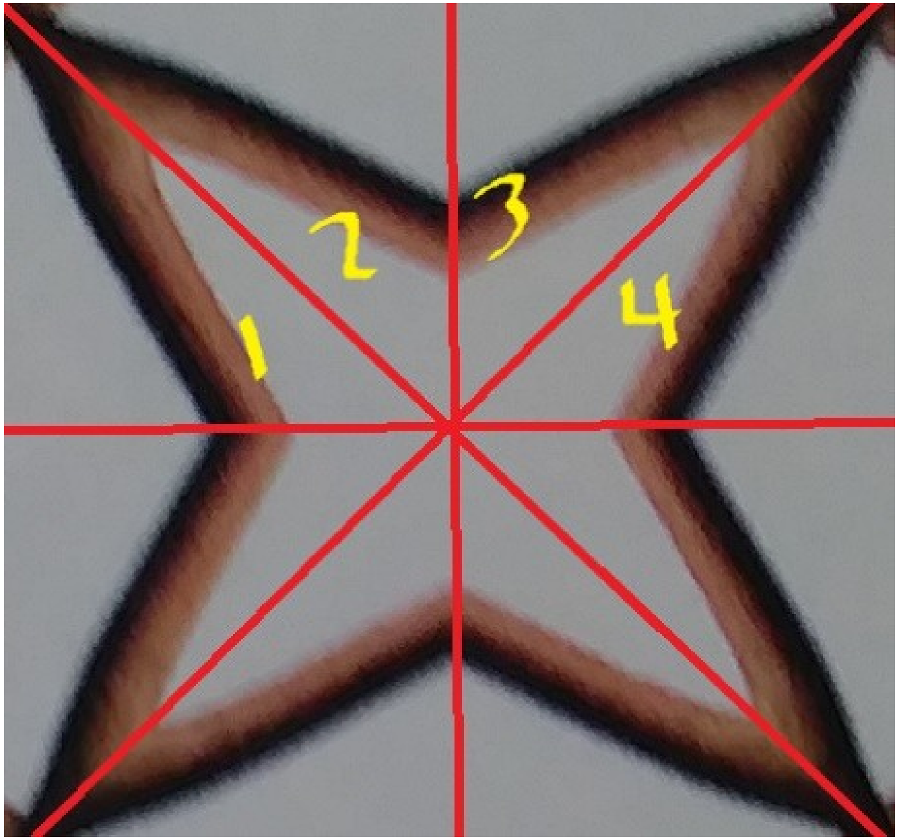
Observe that this pattern also has reflection symmetry

Consider the 4-sided diamond



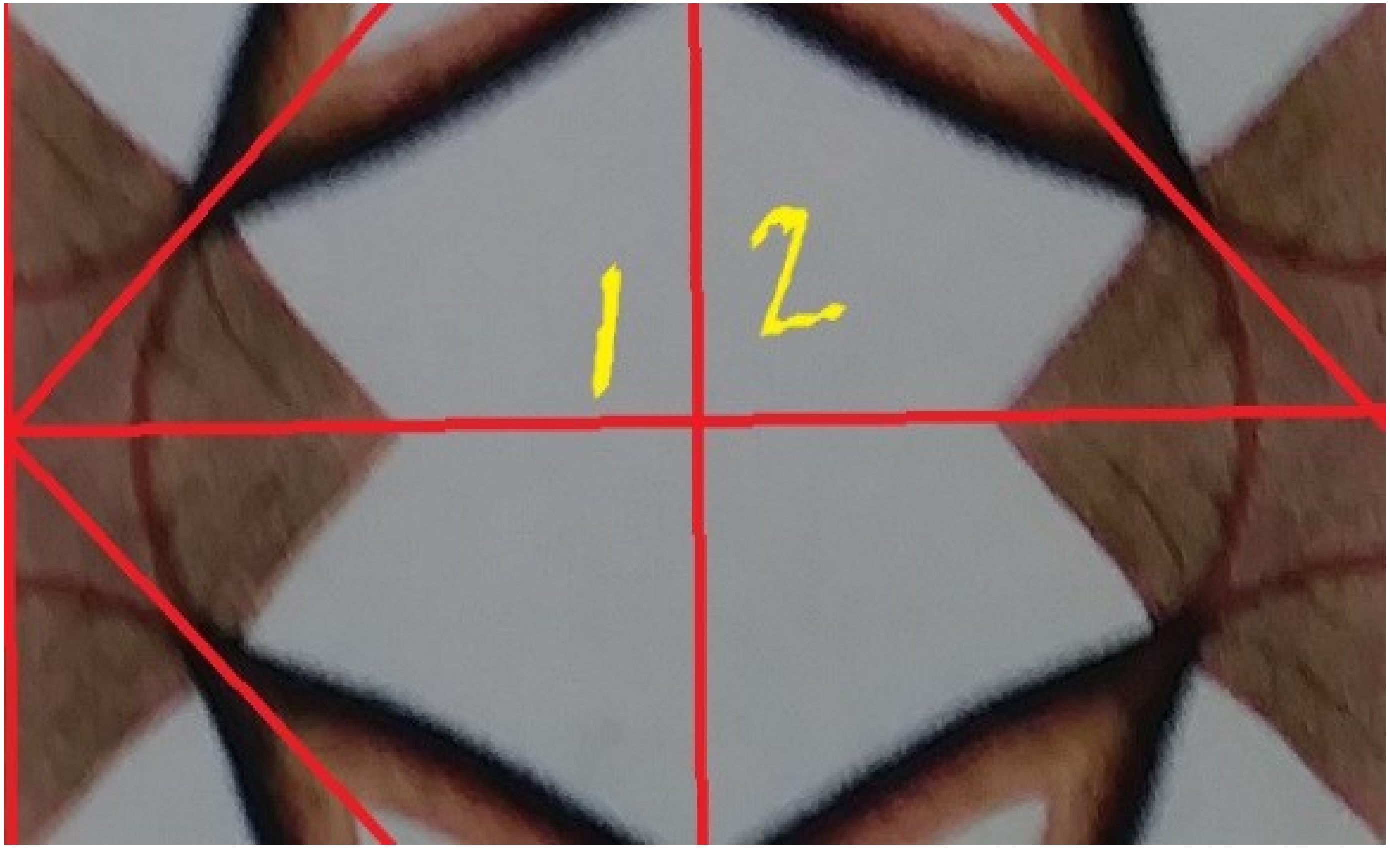
There are 4 unique lines of reflection intersecting at a point

Consider the 4-triangle star



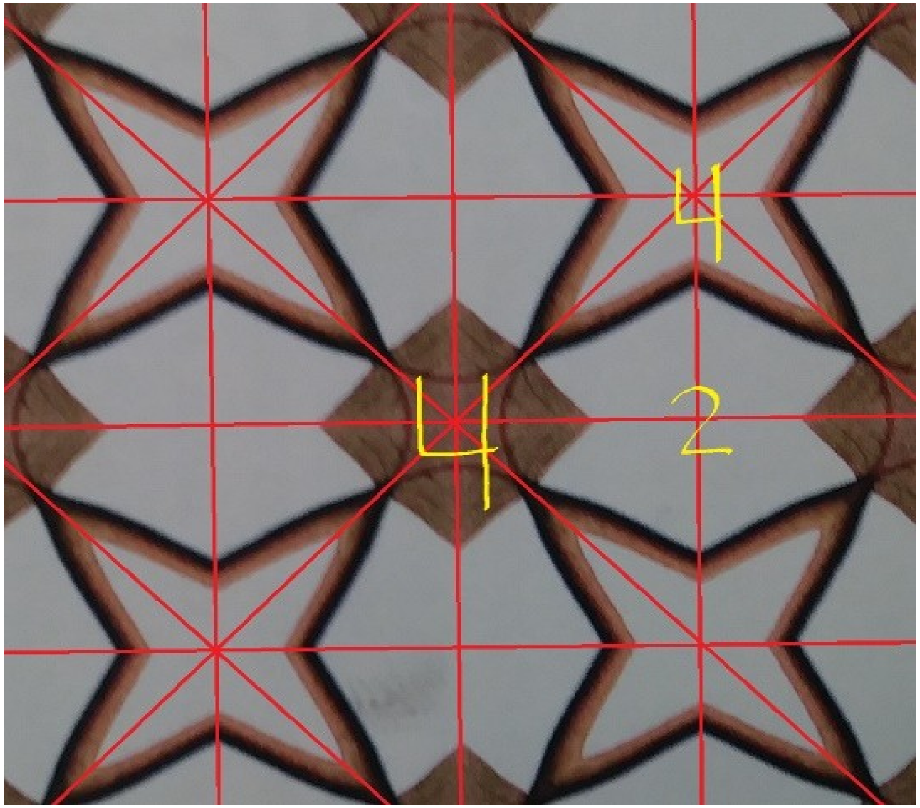
There are 4 unique lines of reflection intersecting at a different point

Consider the other region that looks like empty space



There are 2 unique lines of reflection intersecting at a point

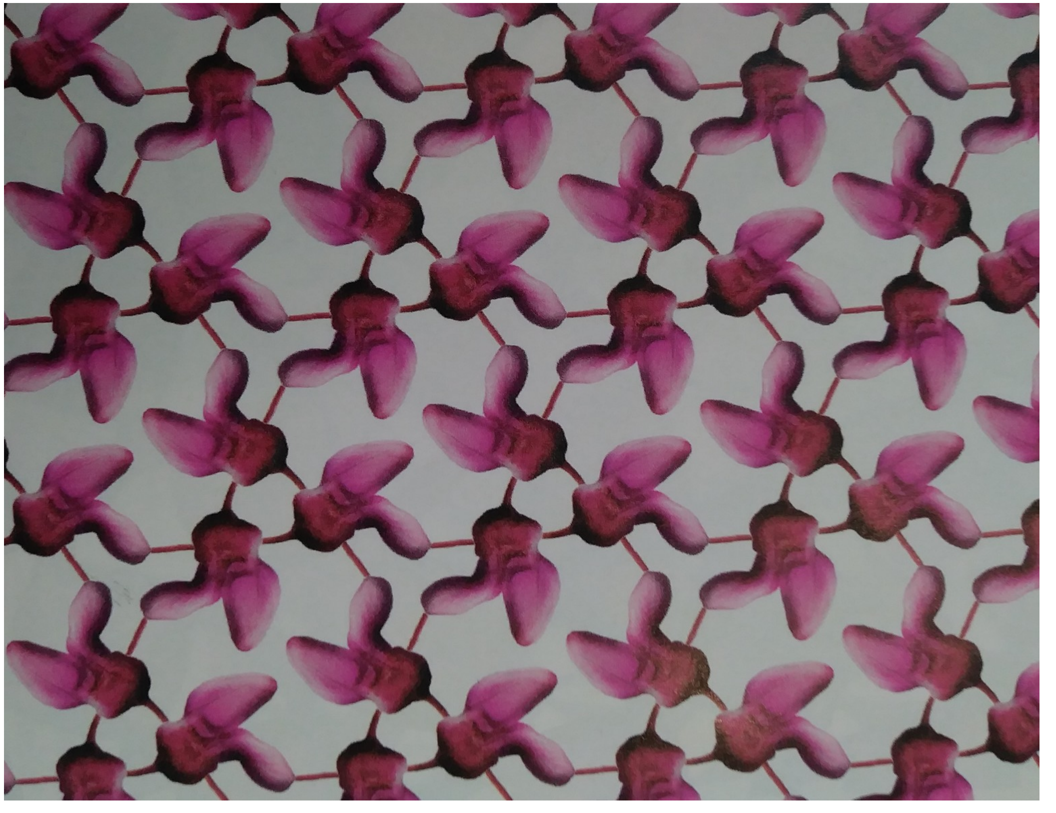
Naming the Signature



Since there are 2 unique points with order 4,

This gets the signature *442

What about Rotations?

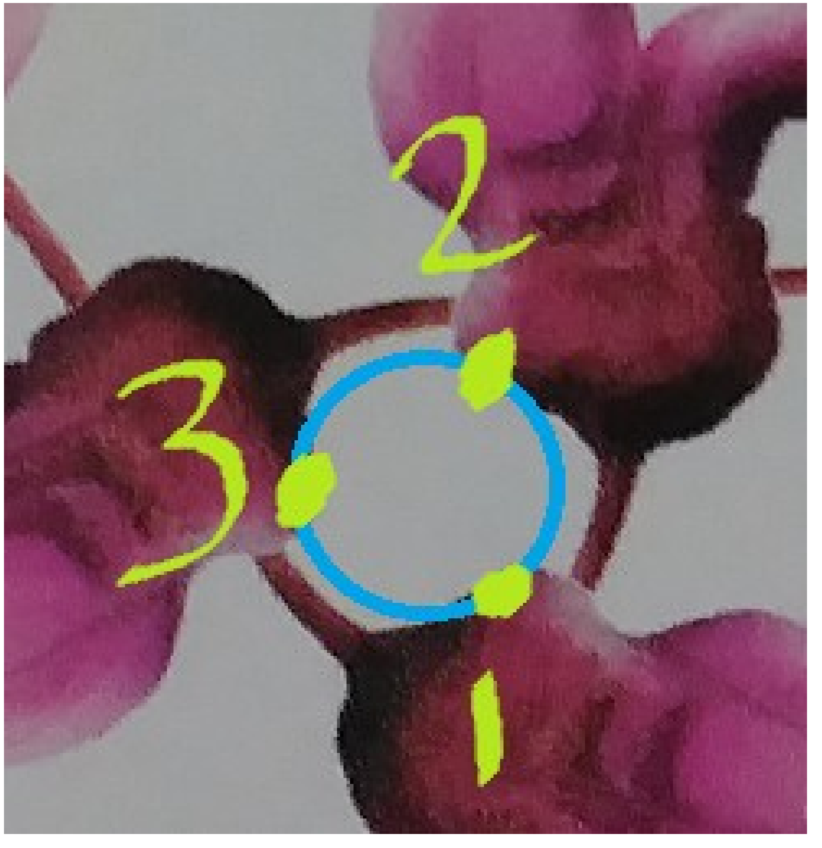


Look for points that are rotated about a circle

Find the points that are identical

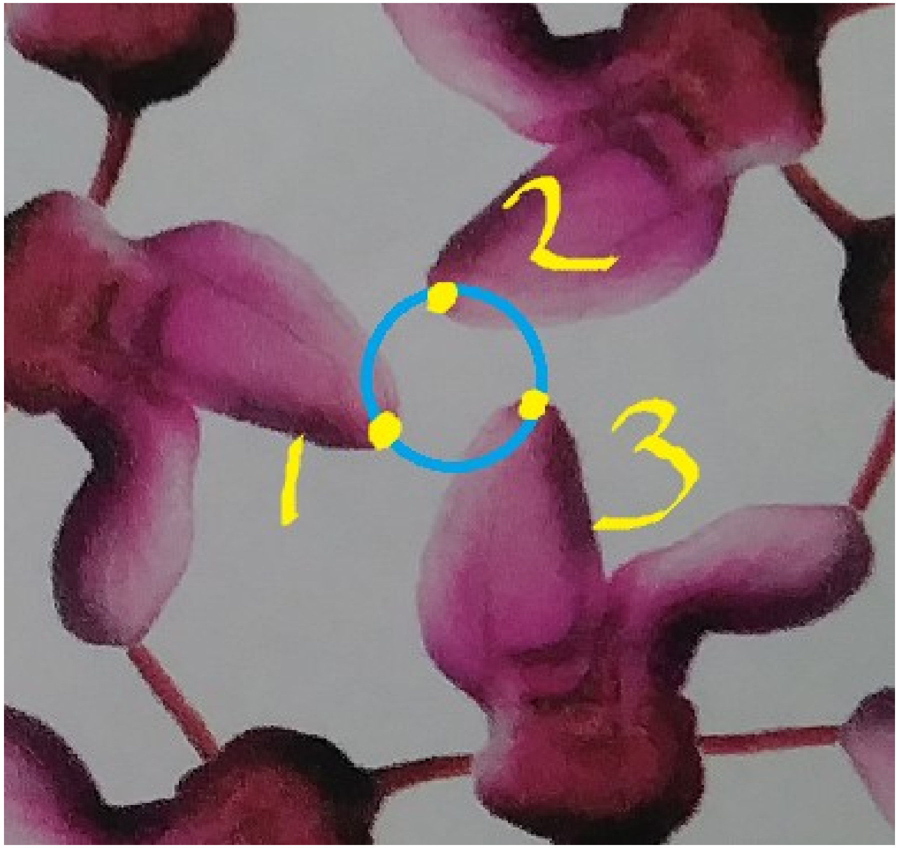
This is a rotation of order 3, because there are 3 identical points

Find the points that are identical



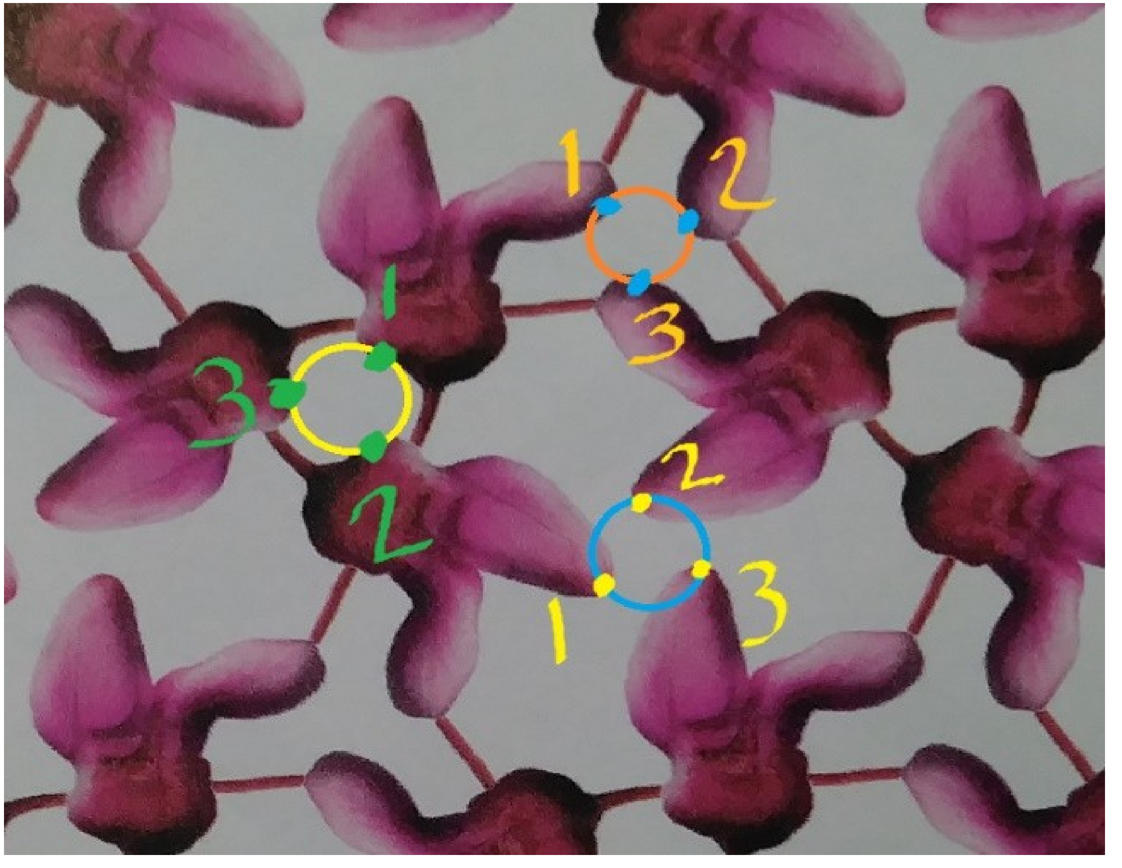
This is a different rotation of order 3, because there are 3 identical points rotated around a different center

Find the points that are identical



This is a different rotation of order 3, because there are 3 identical points rotated around a different center

Naming the Signature



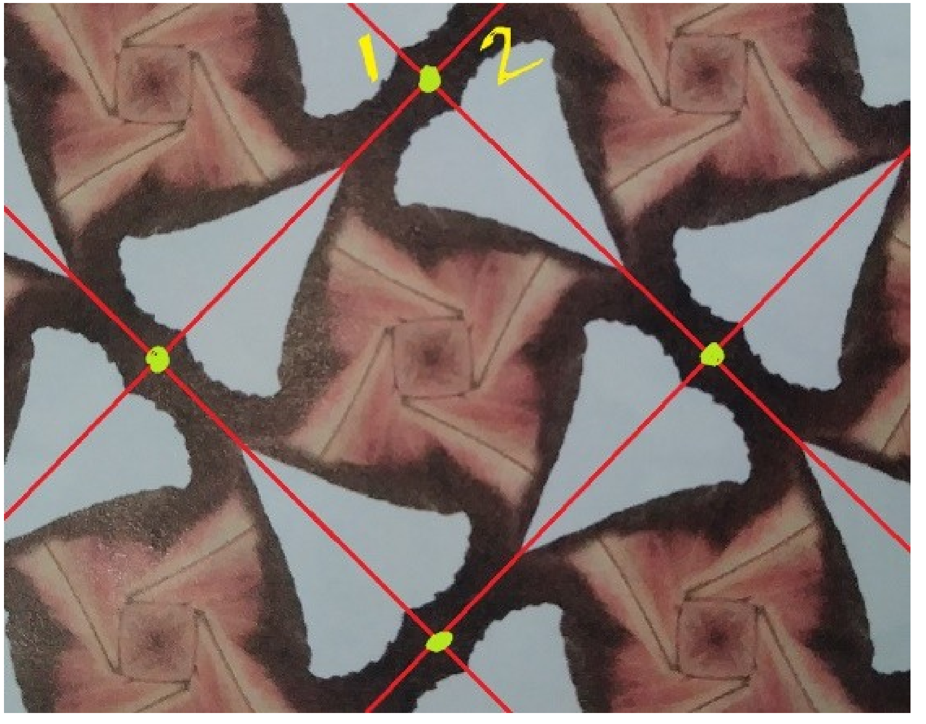
Since there are 3 different points with rotation order 3,

The signature is 333

What if there's both?

There's a rotation and reflection in this picture

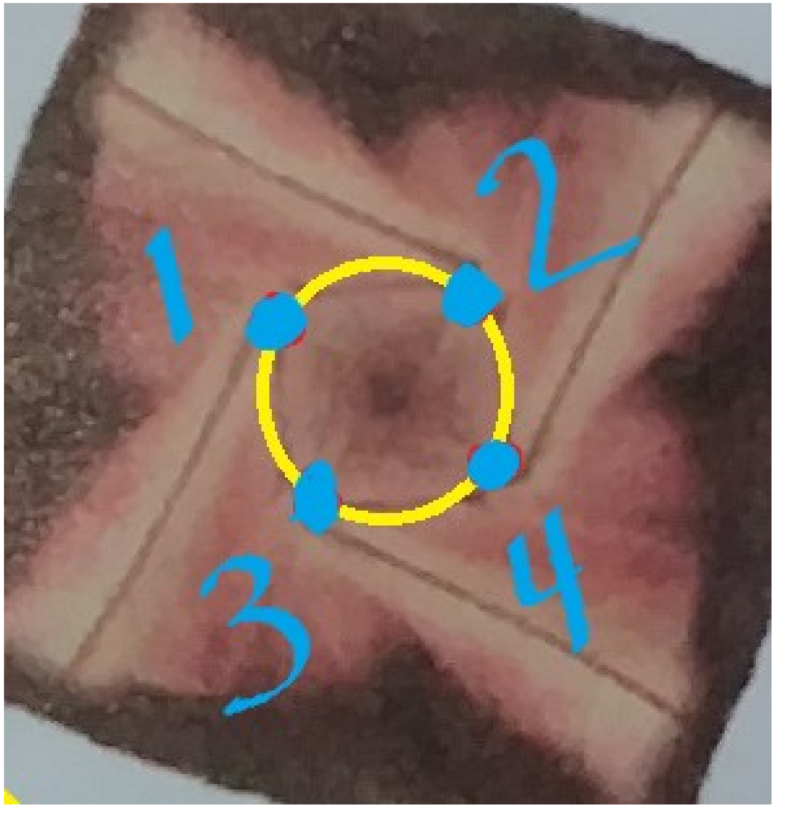
Find the reflection



The 4 green points are all the same point. They either are exactly the same or just rotated, so they are not unique. When we write the signature, we only write the amount of reflections once.

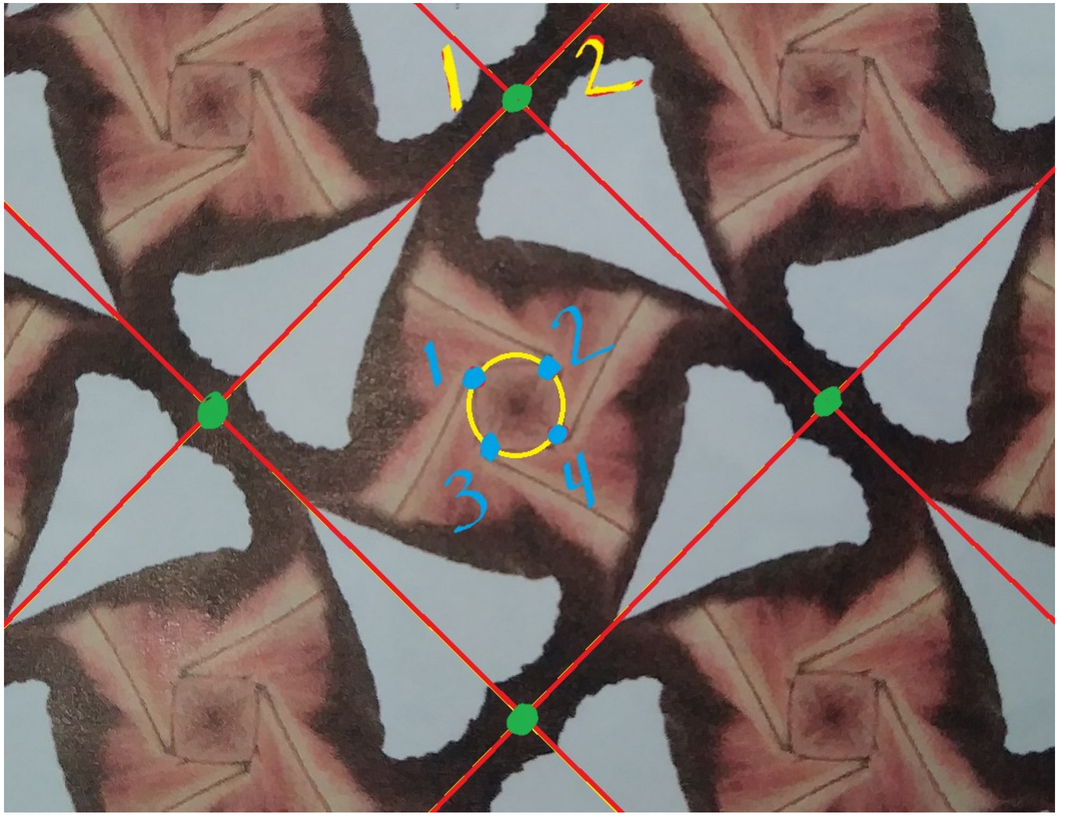
There are two lines of reflection that intersect at a point

Find the rotation



There are 4 points in the square that are identical on rotation so this is rotation of order 4

Naming the Signature



The signature is 4*2

Different Kinds of Signatures

- Some patterns will not have intersecting lines of reflection

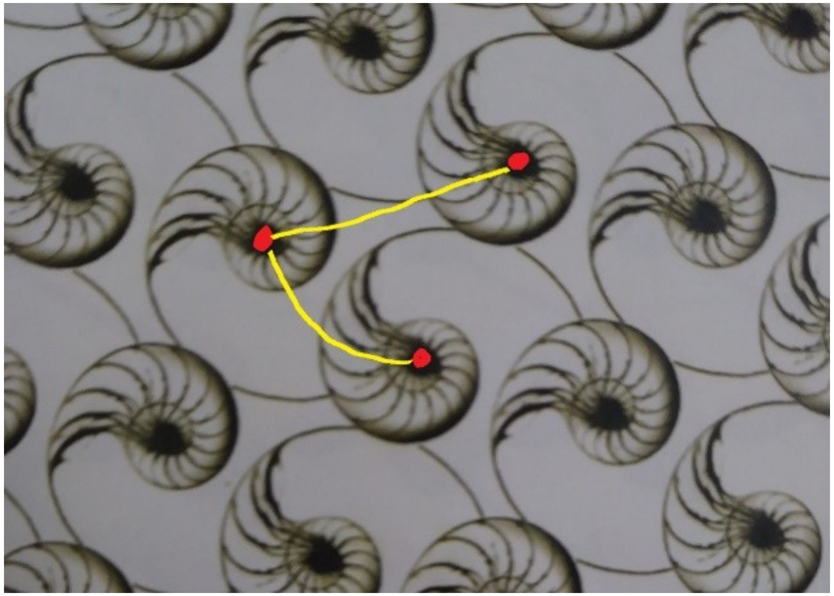
- Some patterns will not have reflections or rotations. These patterns have different types of symmetries

Reflections that don’t intersect

This pattern has 2 distinct lines of reflection. It gets the signature * *

Patterns that change orientation, without reflection

- This pattern has spirals that have different orientations, meaning there is some sort of reflection going on between the spirals, but we cannot draw a line of reflection.
- This is called a miracle, and has the notation ×

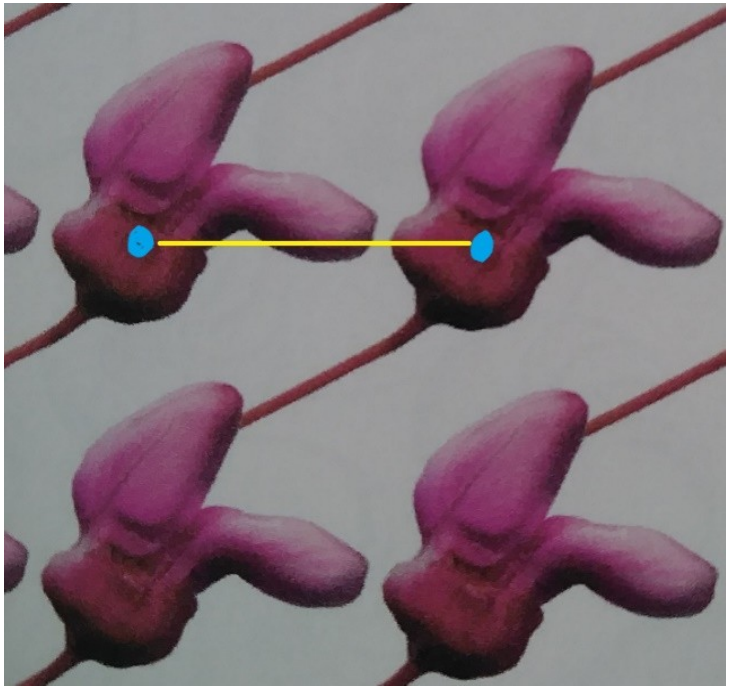


There are two such paths, therefore the signature is

××

Symmetries that do not change orientation, but cannot be formed by rotation

This pattern has points that are identical but cannot be described by a reflection or rotation. This is called a wonder, and it will involve some sort of translation.



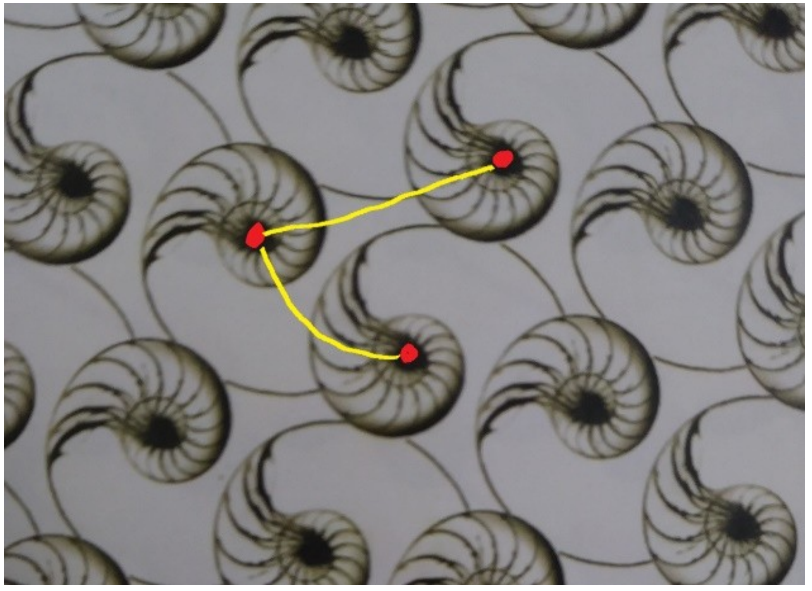
Wonders have signature ₵

How many ways can we generate symmetry on a plane?

- The Magic Theorem says there are only 17!
- To prove that there are only 17, let us assume that the Magic Theorem is true, which we will prove later.
- Assuming the theorem is true, we use a cost function and table provided to create equations that generate the signatures

Reflections and Miracles

- Miracles are described as a sort of reflection in that they change orientation but do not pass over a reflection line



- Reflections can be miracles if the composed action is nothing, so any singular reflection * can be replaced by a miracle ×

Magic Theorem: Cost Table

Symbol	Cost ($)	Symbol	Cost ($)
∘	2	$*$ or $\times$	1
2	$\frac{1}{2}$	2	$\frac{1}{4}$
3	$\frac{2}{3}$	3	$\frac{1}{3}$
4	$\frac{3}{4}$	4	$\frac{3}{8}$
5	$\frac{4}{5}$	5	$\frac{2}{5}$
6	$\frac{5}{6}$	6	$\frac{5}{12}$
⋮	⋮	⋮	⋮
N	$\frac{N-1}{N}$	N	$\frac{N-1}{2N}$
∞	1	∞	$\frac{1}{2}$

Table 3.1. Costs of symbols in signatures.

Creating Equations

- Start by creating the signatures purely made out of symbols.

₵ - the wonder is automatically assigned cost=2

× and * both have cost =1, so

**

*×   all have cost = 2

××

Creating Equations
Total cost of signature = 2

Rotations

$$\sum \frac{N-1}{N} = 2$$

Reflections

All Reflections begin with * which has cost 1, therefore we can extract this 1 and write it as an integer

$$\left(\sum \frac{N-1}{2N}\right) + 1 = 2$$

$$\frac{1}{2}\left(\sum \frac{N-1}{N}\right) = 1$$

$$\sum \frac{N-1}{N} = 2$$

Since this is exactly the same as the equation for rotation symmetries, we can say that any signature of the form XYZ has a matching *XYZ

Creating Equations

$$\sum_{i=1}^{k} \frac{N_i - 1}{N_i} = 2$$

becomes

$$\sum_{i=1}^{k} \frac{N_i}{N_i} - \sum_{i=1}^{k} \frac{1}{N_i} = 2$$

becomes

$$k - 2 = \sum_{i=1}^{k} \frac{1}{N_i}$$

Creating Equations

Because $k-2=\sum_{i=1}^{k}\frac{1}{N_i}$, it follows that $k \geq 3$.

This follows for two reasons, namely

1) The length of the signature, represented by k, must be a positive counting number
2) If k=2, then the sum must = 0, which is impossible

Finding bounds for k

If k = 3,

then $1 = \frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3}$, which allows us to find three numbers whose reciprocals add to 1

If k = 4,

then $2 = \frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} + \frac{1}{N_4}$, which allows us to find four numbers whose reciprocals add up to 1

Since we required N to be ≥ 2, we can quickly see that all the N terms must = 2 for this equation to hold

If k = 5,

then $3 = \frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} + \frac{1}{N_4} + \frac{1}{N_5}$

However, since N must be at least 2, we can quickly see that the left side is greater than the right, or that 3 > 2.5

We can also see that as k increases, the left side will increase by 1, and the right by ½, meaning that we cannot have a signature longer than 4 characters

Solving for N

We are left with the equations

$$1 = \frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3}$$

$$2 = \frac{1}{N_1} + \frac{1}{N_2} + \frac{1}{N_3} + \frac{1}{N_4}$$

which, after some work give us the signatures

333 and *333

632 and *632

442 and *442

2222 and *2222

Keeping Track

1. ¢
2. **
3. *×
4. ××
5. 333
6. *333
7. 632
8. *632
9. 442
10. *442
11. 2222
12. *2222

Going from 12 to 17

Where do the other 5 generating signatures come from?

Based on the cost formula, rotations have cost $\frac{N-1}{N}$

And reflections have cost $\left(\frac{N-1}{2N}\right)$

Using the costs described in the table, we can set up the equality

$$\frac{N-1}{N} = 2\left(\frac{N-1}{2N}\right)$$

Because this equality exists, it is possible to substitute two reflections of some order for a rotation of the same, therefore

n* = *nn

Going from 12 to 17

By the formula substitution

*333 ✉ 3*3

*442 ✉ 4*2

*2222 ✉ 2*22 ✉ 22*

By the idea of converting a single * into a ×,

22* ✉ 22×

Now counting…17 different types

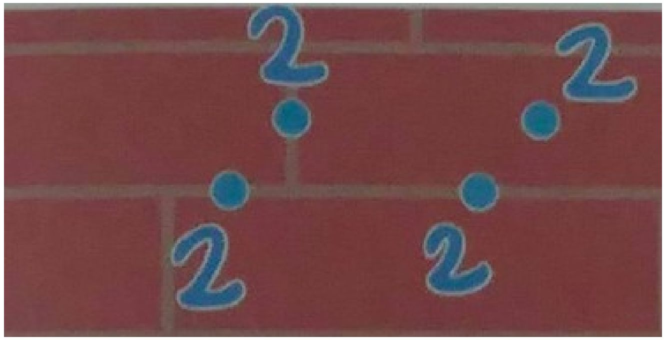
1. ¢
2. **
3. *×
4. ××
5. 333
6. *333
7. 632
8. *632
9. 442
10. *442
11. 2222
12. *2222
13. 3*3
14. 4*2
15. 2*22
16. 22*
17. 22×

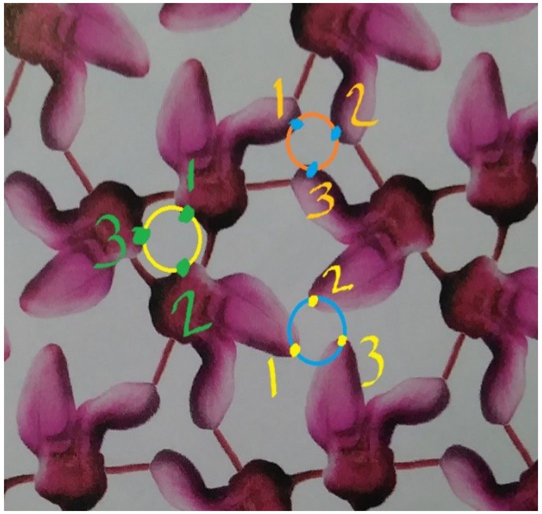
or

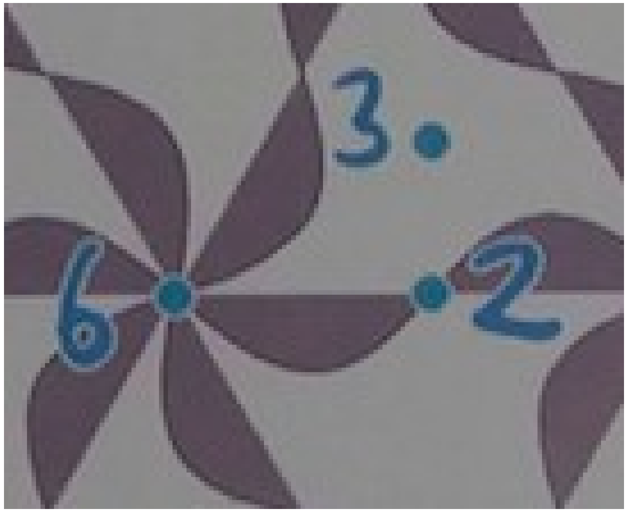
Or, in a more organized presentation

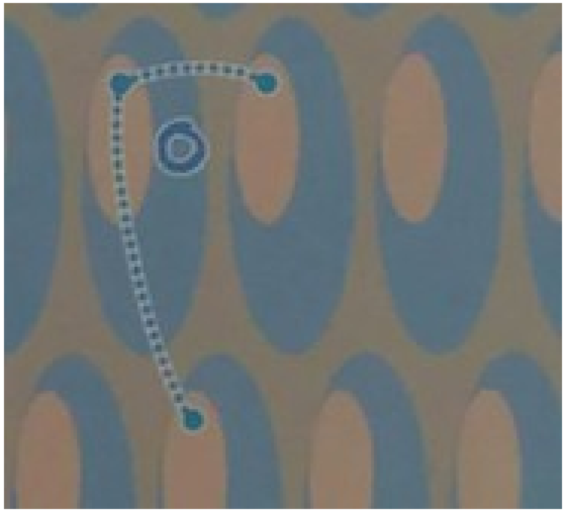
○	632	333	442	2222
				22×
		3*3	4*2	22*
				2*22
** → *× → ××	*632	*333	*442	*2222

Rotation Only Patterns (Examples)

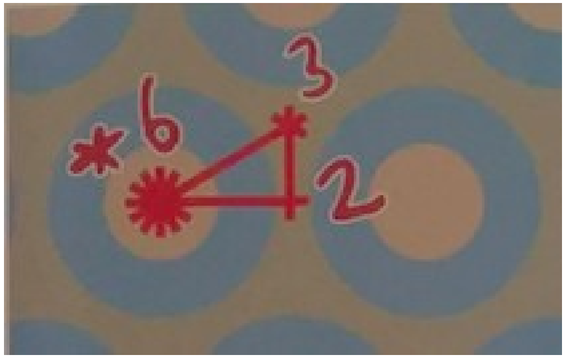


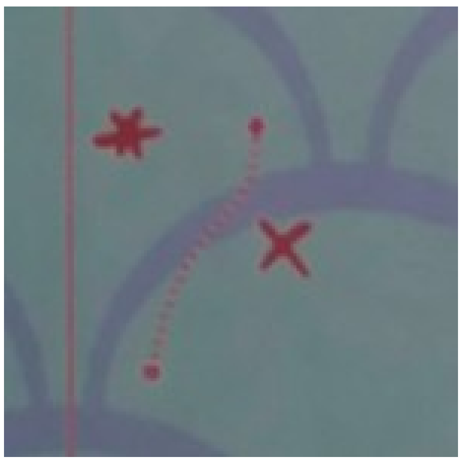


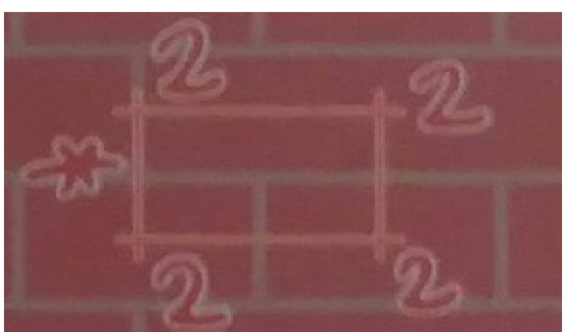


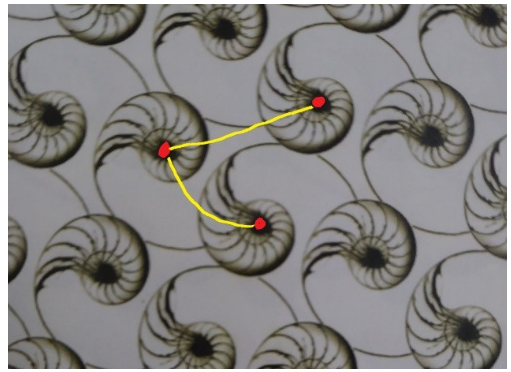


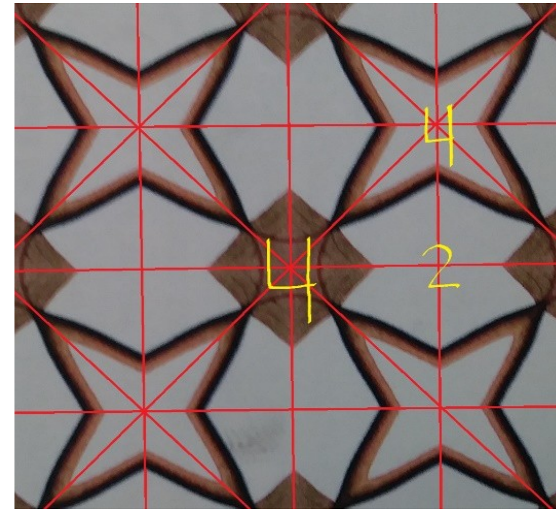
Reflection only

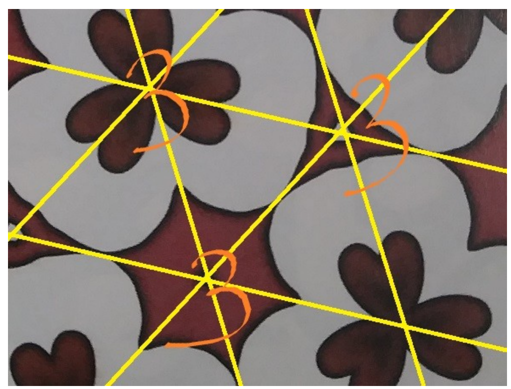












Hybrid Types

The Magic Theorem

There are only 17 ways to generate symmetry on a plane.

To prove this, we will need to use the Euler Characteristic formula

$$V - E + F = 2$$

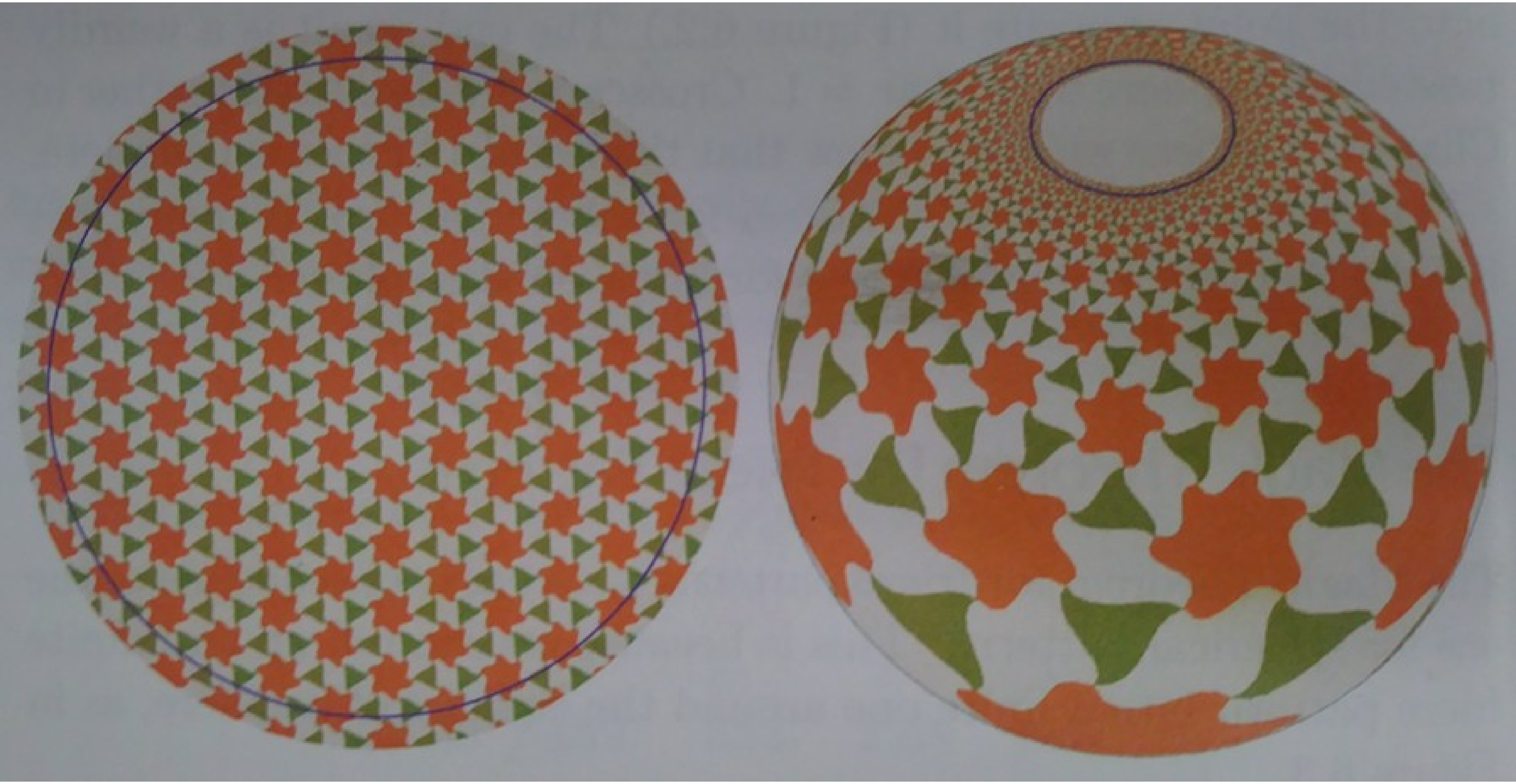
Proving the Magic Theorem

Prove: An orbifold corresponding to a Euclidean plane has "change" = 0

"change" is given by the formula $v - e + f = 0$

Steps and Lemmas

1) The number of vertices, edges, and faces inside the circular region is proportional to the area of the circle
2) The number of vertices, hedges, and faces cut off by the boundary of a region is proportional to the boundary



Proving the Magic Theorem

Proving Lemma 1

Let there be a circle P with radius R

|P| = area of P = πR^2

Let Q be a region inside P with constant area

|Q| = area of Q = A

If we assume there are V vertices inside P and v vertices inside Q, we can argue

$$\frac{v}{|Q|} \sim \frac{V}{|P|} \quad or \quad \frac{v}{\mathrm{A}} \sim \frac{V}{\pi \mathrm{R}^2}$$

By multiplication, we can say

$$\frac{\pi \mathrm{R}^2 v}{\mathrm{A}} \sim V$$

or

$$kR^2 v \sim V$$

when $\mathrm{k} = \frac{\pi}{A}$

Proving the Magic Theorem

Proving Lemma 2

Let Q_N be the n copies that cover P

The area of Q_n – the area of P is proportional to the circumference, or $|Q_N| - |P| \cong c(2\pi\mathrm{R})$

From Lemma 1,

$|Q| = \alpha(v)$, which implies that $|Q_N| = N * \alpha(v)$

and

$|P| = \alpha(V)$

It follows that

$$N * \alpha(v) - \alpha(V) = 2\pi c\mathrm{R}$$

and when dividing out α, it follows that

$$Nv - V = \beta R$$

Proving the Magic Theorem

Proving the Theorem

Assume there are N copies. By our proof of Lemma 1

$$\mathrm{N} = k\mathrm{R}^2$$

If there are N copies, then we have Nv vertices inside our copies of Q.

Assume that $(\mathrm{N}v - V)$ approximates the number of vertices inside the N copies of Q and outside the circle P.

By our proof of Lemma 2, this will be proportional to βR.

$$(\mathrm{N}v - V) < \beta\mathrm{R}$$

By passing the edges and faces through the same lemmas, it is possible to arrive at the conclusion that

$$(\mathrm{N}e - E) < \beta\mathrm{R}$$

$$(\mathrm{N}f - F) < \beta\mathrm{R}$$

Proving the Magic Theorem

Therefore,

$$(\mathrm{N}v - V) - (\mathrm{N}e - E) + (\mathrm{N}f - F) < c\mathrm{R}$$

where c is proportional to β, and it follows that

$$Nv - Ne + \mathrm{N}f - V + E - F < c\mathrm{R}$$

$$Nv - Ne + \mathrm{N}f < V - E + F + c\mathrm{R}$$

$$v - e + f < \frac{V - E + F + c\mathrm{R}}{N}$$

Using the Euler Characteristic here, this equation becomes

$$v - e + f < \frac{2 + c\mathrm{R}}{N}$$

$$v - e + f < \frac{2 + c\mathrm{R}}{k\mathrm{R}^2}$$

Proving the Magic Theorem

If we let char = Euler Characteristic = $V - E + F = 2$,
and we let ch = change subtracted from cost = $v - e + f$
proving that ch = $v - e + f = 0$ proves the Theorem.

$$\text{ch} = v - e + f < \frac{2 + c\mathrm{R}}{k\mathrm{R}^2}$$

or simply

$$\text{ch} < \frac{2 + c\mathrm{R}}{k\mathrm{R}^2}$$

Proving the Magic Theorem

As R gets larger, the right side tends to 0, so we have an upper bound of 0 for ch.
To finish the proof, it is necessary to prove a lower bound of also 0.
To do this, consider the Euler Characteristic, and the orbifold that we are describing using v, e, and f

v is a fraction of V, e is the same fraction of E, f is the same fraction of F
Therefore, it follows that for some positive constant c,

$$v - e + f = \frac{V - E + F}{c} > 0$$

or simply

$$v - e + f = \frac{2}{c} > 0$$

which, when c is positive, which it must always be, requires

$$v - e + f > 0$$

If $v - e + f$ has an upper and lower bound of 0, then it must be 0.
It follows that ch = 0, and cost is 2, as the formulas suggest,
and then theorem is proven