<u>**Counting and Burnside's Lemma**</u>

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Abstract

An important problem in mathematics, which arises in probability, chemistry, and other fields, is listing and counting patterns. As a simple example, consider a square where each edge can be either black or white, represented by a sequence such as BBBB or BWWB. If the patterns are all considered different, there are 16 total. However, if patterns are considered equivalent when the square is rotated or reflected, there are only 6 different arrangements: BBBB, BBBW, BBWW, BWBW, WWWB, WWWW. If the total number of patterns is small, the number can be found by listing, but when the patterns are more complex, for example by increasing the number of colors, or by changing the meaning of "equivalent", mathematical tools are needed. Burnside's Lemma from Group theory (the mathematics of symmetry) gives a fundamental counting method: it says that the number of different arrangements is the average number of patterns unchanged by each of the allowable "moves" (such as rotation). In this project, I illustrate counting using some basic arrangements, then expand counting to infinite frieze patterns, and explain how Burnside's Lemma can be used to count them. I also discuss how and why Burnside's Lemma works.

Table of Contents

Introduction

In its most general definition, counting is a branch of mathematics whose goal is to determine the number of elements in a set, or in other words, determine the number of possibilities, given certain criteria. As a mathematical subject, counting focuses not on counting a particular set (or solving a particular counting problem), but rather on finding patterns in order to solve entire classes of counting problems.

As a basic example, consider a square divided into 4 square cells in a 2×2 arrangement, where each cell can be either black or white [see Figure 1]. In this case, the number of possible arrangements can be found by simply listing them all, but a more general way to consider this situation is to notice that for each cell, any decision made about its color does not affect (in particular, it does not limit) the choices that can be made for any other cell. These four decisions can be made independently, so the total number of possibilities is simply the product of the number of options for each choice: there are four choices, and two options for each, so there are $2\times2\times2\times2 = 2^4 = 16$ possible arrangements. In general, there are n^m possibilities, where n is the number of colors, and m is the number of cells.

Counting and Equivalence

If someone were drawing the arrangements in Figure 1, each on a separate piece of paper, and wanted to know how many pieces of paper must be used in order to represent all possible arrangements, then they would not need to construct all 16 of them. Many arrangements are redundant – if rotating an arrangement by 180° produces another arrangement, then only one of those two will need to be constructed, since either one of them can represent both. Similarly, if rotations by other amounts, reflections, or any other movement link two or more arrangements, those may also be considered equivalent (not "truly unique"). These movements are symmetries (where the overall position does not change), or more generally, permutations, when the objects are not necessarily geometric (permutations map a set to itself). Exactly which arrangements are equivalent can be defined by which of these symmetries/permutations are allowed, and this can vary greatly, even within a single set of arrangements – we need not always allow all possible rotations, reflections, and other symmetries.

In the previous 2×2 case, suppose equivalence is defined by saying that if an arrangement can be produced by rotating another arrangement by a multiple of 180° clockwise, then those two arrangements are considered equivalent. The collection of selected symmetries would be 180° clockwise rotation, and 0° clockwise rotation (equivalent to a 360° rotation). Instead of 16 arrangements, there are now only 10 different ones [Figure 2]. If instead, horizontal reflections, vertical reflections, the two diagonal reflections, and multiples of 90° rotations are all allowed, then there would be 6 different arrangements [Figure 3]. In general, as the collection of symmetries gets larger, the number of different arrangements will decrease, as there are more potential ways for any two arrangements to be equivalent. After consideration of equivalence, each truly different element represents an equivalence class wherein each arrangement in that class is equivalent to every other arrangement in that class, and is not equivalent to any arrangement not in that class. These equivalence classes are also called orbits. More formally, if A is a selected symmetry and i is a particular arrangement, then A(i) is equivalent to i. The set of selected symmetries (possible choices for A) fully define all equivalences for i, so if no allowable symmetry B exists such that B(i) = j, then i is not equivalent to j. The number of truly different elements = the number of equivalence classes = the number of orbits, and this is what we are trying to count.

Equivalence can be defined arbitrarily, even in unexpected and non-intuitive ways. For example, suppose that changing a black cell to a white cell, and changing a white cell to a black cell were selected symmetries. Any of those 16 arrangement can be converted to any other one by repeatedly changing the color of cell, so they are all equivalent, and there is only one unique arrangement. To ensure that the set of symmetries gives well-defined equivalence classes, we will assume the set forms a group G acting on the set of arrangements (S) (see [Gallian], p 42).

A group is a set with a binary operation (×) such that the four axioms below are true. In a symmetry or permutation group, the binary operation is function composition:

- Identity: $\exists$ an element e $\in$ G such that $\forall$ A $\in$ G, e × A = A × e = A.
- Closure: $\forall$ A and B $\in$ G, we have (A × B) $\in$ G.
- Inverses: $\forall$ A $\in$ G, $\exists$ a $A^{-1} \in$ G such that $A \times A^{-1} = A^{-1} \times A = e$.
- Associativity: $\forall$ A, B, and C $\in$ G, (A × B) × C = A × (B × C).

Although these properties do not arise specifically from counting, they are intuitively desirable for sets of symmetries defining equivalence. The identity must be present because if it is not, then the claim is made that an element with nothing done to it is not equivalent to that element (in other words, $i \neq i$). If closure did not hold, a contradiction would be possible: suppose that clockwise rotation by 90° was a selected symmetry but rotation by 180° was not, then we could have j = R_{90}(i) and k = R_{90}(j) = $R_{90}R_{90}$ (i) = R_{180}(i), so i would be equivalent to j and j to k, but i would not be equivalent to k.

If these axioms are met, then we can prove that a group of symmetries, or, more generally, a group of permutations, defines an equivalence relation, as stated in the following lemma:

<u>Lemma</u>: Let G be a group acting on a set S (a group of permutations of S), and define the following relation on S: $i \sim j$ if A(i) = j for some A in G (j is in the orbit of i);

then ~ is an equivalence relation, which is a binary operation that is reflexive, symmetric, and transitive.

<u>Proof</u>: Clearly, ~ is a binary relation, as it relates two elements.

Reflexivity: ~ is reflexive if for all i in S, $i \sim i$. $i \sim i$ if there exists some A in G such that A(i) = i. By the group axioms, G must contain an identity element. Let A = e, the identity element. So A(i) = e(i) = i, so $i \sim i$, so ~ is reflexive.

Symmetry: ~ is symmetric if for all i and j in S, $i \sim j$ implies $j \sim i$. Suppose $i \sim j$. Then there exists some A in G such that A(i) = j. By the group axiom for inverses, for each A in G, there exists an element A^{-1} in G such that $A^{-1}A = e$. So, $A(i) = j \rightarrow A^{-1}A(i) = A^{-1}(j) \rightarrow A^{-1}(j) = e(i) \rightarrow A^{-1}(j) = i$. Thus, we have found a B in G (namely, A^{-1}) such that B(j) = i, so $j \sim i$.

Transitivity: ~ is transitive if $i \sim j$ and $j \sim k$ imply that $i \sim k$. Suppose $i \sim j$ and $j \sim k$. Then there exists some A, B in G such that A(*i*) = *j* and B(*j*) = *k*. So B(A(*i*)) = *k*. By the group axiom for closure, for all A, B in G, B × A is also in G. So let C = B × A → C(*i*) = *k*, so *i* ~ *k*. ■

To apply the lemma above to counting, we prove the following lemma on orbits (equivalence classes).

<u>Theorem</u>: The orbits of S constitute a partition of S, that is, each element in S appears in exactly one orbit. Moreover, for all *i*, *j* in S, if *j* is in the orbit of *i*, then orb(*i*) = orb(*j*).

<u>Proof:</u> [If *j* is in the orbit of *i*, then orb(*i*) = orb(*j*)] Let G be a group acting on a set S and let *i*, *j* be members of S. Suppose that *j* is in the orbit of *i*, so there exists some A in G such that A(*i*) = *j*. We will prove that orb(*i*) = orb(*j*) by proving that each orbit contains the other. Let *p* be a member of orb(*i*) → *p* = B(*i*) for some B in G. A(*i*) = *j* → *i* = $A^{-1}(j)$. Therefore, *p* = B(*i*) = $B(A^{-1}(j)) = BA^{-1}(j)$. BA^{-1} is in G by the closure axiom, so *p* is in orb(*j*). Thus, orb(*i*) is contained within orb(*j*). By a similar proof, orb(*j*) is contained within orb(*i*). Because orb(*i*) and orb(*j*) each contain each other, orb(*i*) = orb(*j*) whenever *j* is in the orbit of *i*.

[Orbits constitute a partition of S] We want to show that each element in S appears in exactly one orbit. Let *r* be some element in S. Since *r* = e(*r*), *r* is a member of orb(*r*), so *r* appears in at least one orbit. Suppose *r* is a member of orb(*t*) and orb(*u*). Then, X(*t*) = *r* = Y(*u*) for some X and Y in G → $Y^{-1}X(t) = Y^{-1}Y(u) \rightarrow Y^{-1}X(t) = e(u) \rightarrow u = Y^{-1}X(t)$. $Y^{-1}X$ is in G by the closure axiom, so *t* is in orb(*u*). From above, orb(*t*) = orb(*u*), and therefore *r* cannot be the member of two distinct orbits, so it is a member of at most one orbit. We have shown that *r* is a member of at least one orbit and at most one orbit, so all elements in S appear in exactly one orbit. That is, the orbits of S constitute a partition of S. ■

Frieze Patterns and Burnside's Lemma

This section is based on the article [Pisanski, et al].

We are not limited to counting only finite arrangements; infinite arrangements can also be counted if they have some structure or pattern that limits the number of possible arrangements to some finite amount. Frieze (strip) patterns are such a type of arrangement. They are constructed by infinitely repeating an element along a single axis so that it will have translational symmetry along that axis [Figure 4]. The complexity of frieze patterns can vary greatly, based mostly on how complex the repeated element is. In order to provide some structure for considering such a diverse class of arrangements, we will consider the element that is being repeated as being constructed of a row of identically-sized, decorated rectangles. The rectangles are formed by creating an (asymmetric) pattern on one rectangle, then applying symmetry operations (rotation and/or reflection) to form up to four possible aspects. The dimensions of this row ($1 \times n$) will vary.

The pool of allowable aspects and the symmetry group will also vary. For example, consider all frieze patterns that could be produced by repeating a 1×2 element consisting of two possible aspects, one pointing northwest, and one pointing southeast. The choices made for any position on the element do not limit or affect the choices available for any other position on the element, so each position has two independent choices, so there are $2^2 = 4$ possible elements that can be produced, so there are 4 possible infinite patterns that can be produced by repeating them [see Figure 4]. But we must also consider symmetries and equivalence. For frieze patterns (oriented so the axis is horizontal), only certain types of symmetries are possible: translation, 180° rotation, horizontal reflection, and vertical reflection. Equivalence classes in frieze patterns lead to equivalence between the various 1 x *n* elements, but the possible symmetries are determined by equivalence within the infinite pattern, not the 1 x *n* elements [for example, see Figure 6b]. Unlike the previous examples using a simple 2 × 2 grid, translation is a possible symmetry – because frieze patterns are infinite, moving each aspect the same distance along the axis of repetition does not change the overall position or pattern of the strip. Translations by up to *n* aspects are possible symmetries (translations by exactly *n* aspects are equivalent to moving one full element). The only reasonable rotational symmetry is 180 degrees, as a 90 degree rotation will change the overall position of the strip, and that will never be equivalent to the starting arrangement. Reflection across the center axis of translational symmetry, and some reflections perpendicular to the center axis are also possible symmetries. For example, consider the previously mentioned 1 × 2 case, with NW and SE aspects [Figure 4]. There are four possible patterns, and suppose that the selected symmetries are reflection along the horizontal axis, 180 degree rotation, any multiple of translation to the right by one aspect, and any composition of these. Two of the 1 × 2 patterns are equivalent due to translation by one aspect, and the other two are equivalent due to rotation by 180 degrees, so there are two orbits. If the pool of aspects is increased to 4 (NW, SE, NE, SW) [Figure 5], there are 16 arrangements in 4 orbits.

In general, for a 1 × *n* pattern, the possible symmetries will be translations by 0 to *n* – 1 aspects, *n* different glide reflections across the horizontal center line by 0 to *n* – 1 aspects, *n* different rotations by 180° around the center point of any aspect or the line between two aspects, (there are *2n* different centers, but half the rotations are equivalent), and *n* different vertical reflections across vertical lines through the center of an aspect or between two aspects (again, *2n* positions but half the reflections are equivalent) [see Figure 6 for the 1 × 6 case]. All together, there are *4n* symmetries for a 1 x *n* arrangement.

The complexity of the frieze patterns grows rapidly as *n* increases, even with only four aspects. As frieze patterns become more complex, counting them manually by listing each arrangement becomes infeasible, so we would like to have some ways to facilitate this counting. Burnside's Lemma, a counting theorem from group theory, stated below, allows us to find the number of orbits, if we can find the average number of arrangements left unchanged by the group of symmetries.

Definition: Let G be a finite group of permutations of a set S. For all R $\in$ G, fix(R) = {i $\in$ S such that R(i) = i} (the set of arrangements that are unaffected by the permutation R).

Burnside's Lemma: The number of orbits of elements of S under G is

$$(1 / |G|) \times ({}_{R \in G}\Sigma\ |\text{fix}(R)|\ ).$$

For Burnside's Lemma, it is essential to be able to count the number of arrangements/elements fixed (left unchanged) by elements in the group of symmetries/permutations. As a simple example, consider the black and white 2×2 grid in Figure 3. There are 8 symmetries and 16 arrangements [Figure 7]. We want to find the average number of arrangements fixed (left unchanged) by these symmetries. The identity symmetry (rotation by 0°) does nothing, so it fixes all 16 arrangements. 90° and 270° rotations only fix arrangements whose components are all the same color, and there are two such arrangements – all black and all white. 180° rotations exchange components that lie along diagonal lines, so these rotations will fix arrangements where the components diagonally opposite each other are the same color, and there are four such arrangements: all W, all B, first diagonal W and second B, or vice-versa. Reflection across the horizontal axis exchanges components that are on the same side , so this reflection will fix arrangements where each component is the same color as the one above or below it, and there are four such arrangements: all W, all B, left side W and right side B, or vice-versa. A similar argument produces four fixed arrangements for reflection across the vertical axis. Both diagonal reflections exchange the two elements that do not lie along the line of reflection, so they will fix arrangements where those two elements are the same color. The two exchanged elements could be either both black or both white (two options) and the other two elements could be WW, WB, BW, and BB (four options). These choices can be made independently, so there are $2 \times 4 = 8$ arrangements fixed by each of the diagonal reflections. The total number of arrangements fixed is 16 (from the identity) + 2 (R90) + 4 (R180) + 2 (R270) + 4 (H) + 4 (V) + 8 (D) + 8 (D') = 48, which, when divided by 8 (the number of symmetries), gives us 6 as the average number of arrangements fixed by each symmetry. According to Burnside's Lemma, this means that there should be 6 orbits, and by listing each arrangement and manually grouping them into orbits, it is clear that there are indeed 6 orbits. It is worth noting that these fixed arrangements can also be counted by summing the number of symmetries that fix each arrangement (by columns, in Figure 7) rather than summing the arrangements fixed by each symmetry (by rows).

For frieze patterns, although the selected symmetries and their behavior will vary with n, they do have some predictable structure. Suppose that we allow all of the previously mentioned symmetries: translation, rotation, and reflections. In the 1×5 case, the only elements that will be fixed by translation by less than $n = 5$ aspects (and greater than 0) are those in which every one of the 5 aspects is identical, and there are 4 such elements because there are 4 choices of aspect. This can be determined by considering one of the five locations for aspects. Its aspect can be chosen freely. In order for this element to be fixed by translation by one aspect, the next location must also contain that aspect, and the same is true for each of the next 3 locations. Therefore, once a choice is made for one of the locations, it completely limits all choice for any of the others. Because 5 is prime, similar reasoning occurs for translations by 2, 3, and 4 aspects. All elements are fixed by translation by 0 aspects, and there are $4^5 = 1024$ elements in total (disregarding equivalence), so the total number of elements fixed by all translations = [number of elements fixed by translation by 0 aspects] + [number fixed by translation by 1] + … + [number fixed by translation by 4] = 1024 + 4 + 4 + 4 + 4 = 1040.

However, in the 1 × 6 case, there are other elements that can be fixed by translations. Translations by one aspect behave as they do in the 1 × 5 case, so there are also 4 elements fixed by such translations. Using similar reasoning as before, if a choice is made for one of the locations (call this location 1), that element will only be fixed by two-aspect translation if location 3 has the same aspect as location 1, and the next translation requires that location 5 also has the same aspect. However, further two-aspect translations return to location 1, and the pattern repeats, so locations 2, 4, and 6 are not limited by the choice made for locations 1, 3, and 5. If a choice is made for location 2, that choice will similarly determine the aspect that must be at locations 4 and 6. Therefore, for two-aspect rotations, there are $4^2 = 16$ elements that will be fixed, as there are two independent choices, and 4 choices for each. In three-aspect translation, there are three choices, and therefore $4^3 = 64$ fixed elements. Four-aspect translation produces the same numbers as two-aspect, and five-aspect translation behaves like one-aspect. Thus, there are 4096 + 4 + 16 + 64 + 16 + 4 = 4200 elements fixed by all translations.

Consider vertical reflection (along lines perpendicular to the axis of repetition). In both the 1 × 5 and 1 × 6 cases, such reflections can only fix elements when the line lies on the border between two aspects, otherwise, some aspect would have to be equal to its own mirror image. In general, there are *n* possible lines to reflect across that do not pass through any aspects: the left border of each of the *n* aspects (any right border reflection will be equivalent to some left border reflection). These reflections shift aspects by mirroring each aspect, then moving them by placing them on the opposite side of the line, but not changing their distance from the line. Call these reflections F^p, where p = the position of the line, from 0 to *n-1* (left to right). An arrangement will be fixed if each of the aspects is the same as the mirror of the other aspect that shares its distance from the line of reflection. For example, for *n* = 6, consider the arrangement $A^1A^2A^3 \mid A^4A^5A^6$ (| is the line of reflection). This arrangement will be fixed if and only if A^1 = mirror(A^6), A^2 = mirror(A^5), and A^3 = mirror(A^4). In the 1 × 5 case, in order for an arrangement to be fixed by F^0 (| $A^1A^2A^3A^4A^5$), A^3 must be equal to mirror(A^3), which is impossible. Similarly, for F^1 ($A^1 \mid A^2A^3A^4A^5$), in order for an arrangement to be fixed, A^4 must be equal to mirror(A^4). For F^2 ($A^1A^2 \mid A^3A^4A^5$), in order for an arrangement to be fixed, A^5 must be equal to mirror(A^5). F^5 is equivalent to F^0, F^4 is equivalent to F^1, and F^3 is equivalent to F^2, so F^5, F^4, and F^3 will also not fix any arrangements. If *n* is odd, no arrangements will be fixed by vertical reflections, because regardless of where the line of reflection is placed, some aspect will need to be equal to its own mirror, and none of the chosen aspects have this property.

However, in the 1 × 6 case, some arrangements can be fixed by these reflections. F^0 (| $A^1A^2A^3A^4A^5A^6$) will fix arrangements where A^1 = mirror(A^6), A^2 = mirror(A^5), and

A^3 = mirror(A^4) [see Figure 6b]. Therefore, the choice for A^1 determines the choice for A^6 (and similarly for A^2 with A^5, and A^3 with A^4) because in our pool of chosen aspects, given some aspect, there is exactly one aspect that is the mirror of the given one. There are three independent choices to be made for F^0, and each choice has four options, so there are $4^3 = 64$ arrangements fixed by F^0. F^1 ($A^1 \mid A^2A^3A^4A^5A^6$) will produce a different pairing (A^1 determines A^2, A^3 determines A^6, and A^4 determines A^5), but there are still three choices with four options each, so there are also $4^3 = 64$ arrangements fixed by F^1. Similarly, for F^2 ($A^1A^2 \mid A^3A^4A^5A^6$), A^1 determines A^4, A^2 determines A^3, and A^5 determines A^6, so F^2 also fixes 64 arrangements. F^3 ($A^1A^2A^3 \mid A^4A^5A^6$) produces the pairing A^1 with A^6, A^2 with A^5, and A^3 with A^4, and also fixes 64 arrangements, but this equivalent to F^0, so these have already been counted. F^4 is equivalent to F^2, F^5 is equivalent to F^1, and F^6 is equivalent to F^0, so there are $3 \times 64 = 192$ arrangements fixed by all reflections perpendicular to the axis of repetition. In general, if n is odd, no arrangements are fixed by such reflections, and if n is even, then there are $(n / 2) \times 4^{(n/2)}$ arrangements fixed.

Extending these ideas, [Pisanski, et al] show the following. In the 1 × 5 case, 1040 elements are fixed by translation, no elements are fixed by reflection across vertical lines, no elements are fixed by 180° rotations (the counting is similar to that for vertical reflections), and no elements are fixed by glide reflections (horizontal reflection composed with translations). The symmetry group has $4n = 20$ elements, so by Burnside's Lemma, there are 1040 / 20 = 52 orbits.

In the 1 × 6 case, 4200 elements are fixed by translation (shown above), $6/2 \times 4^{(6/2)} = 3 \times 4^3 = 192$ elements are fixed by vertical reflections, another 192 elements are fixed by 180° rotations, and 72 are fixed by glide reflections. By Burnside's Lemma, there are (4200 + 192 +

192 + 72) / 24 = 194 orbits.

By using these patterns and Burnside's Lemma, a formula can be constructed that counts the number of orbits for any *n*, assuming that elements are constructed of the 4 previously discussed aspects, and all possible translation, rotation, and reflections are allowed:

{# fixed by rotations + # fixed by vertical reflections +

fixed by translations + # fixed by glide reflections}/ (4 × *n*) =

{ 2 × [(*n* / 2) × $4^{n/2}$ when n is even; 0 when n is odd] +

[Σ{phi(*k*) × $4^{n/k}$} sum over all positive integers k that divide *n*] +

[Σ{phi(k) × $4^{n/k}$ } sum over all positive, even integers k that divide *n*] }

/ (4 × *n*),

where phi(*k*) [the Euler's phi function] = the number of positive integers less than or equal to *k* that are relatively prime to *k*, for k > 1, and phi(1) = 1.

Proof of Burnside's Lemma

In order to prove Burnside's Lemma, we must first discuss a more fundamental theorem, called the Orbit-Stabilizer Theorem, which will later be used as part of the proof of Burnside's Lemma. The Orbit-Stabilizer Theorem establishes a relationship between the behavior of a group of permutations and the size of orbits (equivalence classes). In counting, the objective is to determine the number of truly unique elements in a set of arrangements. Orbits (equivalence classes) are represented by "truly unique elements", as each orbit contains an arrangement and all of the arrangements equivalent to it, and nothing else. Because orbits partition a set, their size determines how many there will be, and this is exactly what we are trying to determine in counting problems.

Definition: Let G be a finite group of permutations of a set S, orb(i) = {A(i) such that A $\in$ G} (the set of all possible "positions" for i when operated upon by all group elements), and stab(i) = {A $\in$ G such that A(i) = i} (the set of group elements that do not affect i).

Then we have the following theorem:

Orbit-Stabilizer Theorem: For all $i \in$ S, $|\text{orb}(i)| \times |\text{stab}(i)| = |G|$

Discussion: In our use of this theorem, G is the permutation group that defines equivalence, and S is the set of arrangements that are permuted by G. It is worth emphasizing that the equation $|\text{orb}(i)| = |G| / |\text{stab}(i)|$ is always true regardless of which i in S is chosen. For a given group (holding |G| constant), the size of the orbit and the size of the stabilizer are closely linked – if i has twice as large of an orbit compared to another element, then it will have half as many stabilizers.

Proof of Orbit-Stabilizer Theorem

This proof is based on the proof given in [Gallian].

Proof Outline: We will show that stab(*i*) is a subgroup of G, use Lagrange's Theorem to find the number of left cosets of stab(*i*), then produce a one-to-one correspondence between the left cosets and orb(*i*) in order to prove that |orb(*i*)| and |stab(*i*)| have the desired relationship.

Proof: [Show that H is a subgroup of G] Clearly, stab(*i*) = H is a subset of G, since it exclusively contains elements of G which fulfill some additional requirement. We want to prove that H is a subgroup of G, and a subgroup is any subset that is itself a group (that is, it also fulfills the four group axioms).

Associativity holds for all subsets of a group.

By the definition of the identity permutation, e(*j*) = *j* $\forall j \in$ S, so e(*i*) = *i*, so e $\in$ H.

Closure: Let A, B $\in$ H, and let C = AB. C(*i*) = AB(*i*) = A(B(*i*)) = A(*i*) = *i*, so C(*i*) = *i*, so C meets the definition of a stabilizer of *i*, so C $\in$ H, so H is closed.

Inverses: Let A $\in$ H, and let B = A^{-1} (that is, AB = BA = e). A(*i*) = *i* → BA(*i*) = B(*i*) → e(*i*) = B(*i*) → B(*i*) = *i*, so B meets the definition of a stabilizer of *i*, so B $\in$ H.

Thus, H is a subgroup of G.

[Produce a one-to-one correspondence] We want to show that for any *i* in S, the number of cosets equals the size of the orbit (|{AH such that A $\in$ G}| = |orb(*i*)|).

In order to prove that these have equal size, we will construct a one-to-one correspondence between them, so define a mapping T: AH → A(*i*), where A, B $\in$ G. T maps elements in G (the permutation group) to elements in S (the set of arrangements) We want to show that this mapping is one-to-one, onto, and well-defined.

One-to-one: T is one-to-one if whenever two inputs give the same output, then the inputs must have been the same; that is: if T(AH)=T(BH), then AH=BH. Suppose T(AH)=T(BH), then $A(i) = B(i) \rightarrow (A^{-1}A)(i) = (A^{-1}B)(i) \rightarrow (A^{-1}B)(i) = e(i) \rightarrow (A^{-1}B)(i) = i \rightarrow (A^{-1}B) \in H$. By the properties of cosets, AH=BH, so T is one-to-one.

Onto: T is onto if $\forall j \in orb(i)$, then $T(AH) = j$ for some $A \in G$. Suppose $j \in orb(i)$. Then, $D(i) = j$ for some $D \in G$ (by the definition of the orbit)$\rightarrow T(DH) = D(i) = j$, so T on onto.

Well-defined: T is well-defined if when two inputs are equivalent, they will always produce the same output; that is: if AH=BH, then T(AH)=T(BH). Suppose AH=BH, then by the properties of cosets, $(A^{-1}B) \in H$. Since $(A^{-1}B)$ is a stabilizer of i, $(A^{-1}B)(i) = i \rightarrow$

$(AA^{-1}B)(i) = A(i) \rightarrow (eB)(i) = A(i) \rightarrow B(i) = A(i)$, so T is well-defined.

Because T is one-to-one, onto, and well-defined, the set that it maps from and the set that it maps to must be of equal size, so $\forall i \in S$, $|orb(i)| = |\{AH \text{ such that } A \in G\}|$ (the size of the orbit is equal to the number of distinct left cosets of H in G).

Lagrange's Theorem: If G is a finite group and R is a subgroup of G, then |R| divides |G|, and the number of distinct left cosets of R in G is |G| / |R|. Thus, the number of distinct left cosets of H in G is |G|/|H|.

By Lagrange's Theorem, $|\{AH \text{ such that } A \in G\}| = |G| / |H| \rightarrow |orb(i)| = |\{AH \text{ such that } A \in G\}| = |G| / |H|$. H = stab($i$), so $|orb(i)| = |G| / |stab(i)|$, and this is exactly the Orbit-Stabilizer Theorem.

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The Orbit-Stabilizer Theorem provides an indirect way of counting orbits by enabling us to find the size of the orbit containing any particular arrangement by finding the number of permutations that do not affect that arrangement. Determining the size of an orbit is useful because orbits partition the set of arrangements, so if each orbit is larger, there will be fewer of them. However, in counting, we are not usually looking for the size of a particular orbit, but rather the number of orbits in total for the entire set of arrangements. The total number of possible arrangements (disregarding equivalence) is generally not difficult to find, so it would be useful to have a tool that uses the Orbit-Stabilizer Theorem in order to find the average size of orbits, which can be used to readily count them. Burnside's Lemma is such a tool, and it provides a way to directly count the orbits without listing every arrangement or equivalence class.

Definition: Let G be a finite group of permutations of a set S. For all R $\in$ G, fix(R) = {i $\in$ S such that R(i) = i} (the set of arrangements that are unaffected by the permutation i).

Burnside's Lemma: The number of orbits of elements of S under G is

$$(1 / |G|) \times ({}_{R \in G}\Sigma\ |\mathrm{fix}(R)|\ ).$$

Proof of Burnside's Lemma

This proof is based on the proof given in [Gallian] and is illustrated in Figure 7.

Proof Outline: Across all arrangements and all permutations, we will count the pairs of arrangement and permutation for which the arrangement is left unaffected by the permutation. We will count this in two ways (summing stabilizer sets and summing fixed sets) to prove that those quantities are equal. We then use the Orbit-Stabilizer Theorem to prove that the number of stabilizers is equal for arrangements in the same orbit, then use it again to prove that the total number of stabilizers per orbit = |G|. We use this fact to find a relationship between the total number of stabilizers in S and the number of orbits in S. Finally, we use the Orbit-Stabilizer Theorem to find the total number of stabilizers in S.

Proof of Burnside's Lemma:

[Sum of stabilizers = sum of fixes] Consider ordered pairs of the form (R,*i*), where R $\in$ G and *i* $\in$ S (for example, all black and white squares in Figure 7). Let *n* equal the total number of such pairs for which R(*i*) = *i* (the permutation does not affect the arrangement) (the squares outlined in red in Figure 7). Recall that the stabilizers of an arrangement are the permutations that do not affect it, so for any chosen *i* $\in$ S, the number of fixed ordered pairs = |stab(*i*)|. By summing this across all possible *i* $\in$ S (the columns of Figure 7), we obtain *n*: $(_{i \in S}\Sigma\ |\text{stab}(i)|\ ) = n$. Also recall that the fixed set of a permutation are the arrangements that it does not affect, so for any chosen R $\in$ G, the number of fixed ordered pairs = |fix(R)|. By summing this across all possible $R \in$ G (the rows of Figure 7), we also obtain *n*:

$(_{R \in G}\Sigma\ |\text{fix}(R)|\ ) = n$. Therefore, $(_{i \in S}\Sigma\ |\text{stab}(i)|\ ) = n = (_{R \in G}\Sigma\ |\text{fix}(R)|\ )$.

[Number of stabilizers is equal for arrangements in the same orbit] By a theorem proved earlier, the orbits of S constitute a partition of S, that is, each element in S appears in exactly one orbit. Moreover, if *j* is in the orbit of *i*, then orb(*i*) = orb(*j*). Then we have orb(*i*) = orb(*j*) → |orb(*i*)| = |orb(*j*)| → 1 / |orb(*i*)| = 1 / |orb(*j*)| → |G| / |orb(*i*)| = |G| / |orb(*j*)| → |stab(*i*)| = |stab(*j*)|, by the Orbit-Stabilizer Theorem. Therefore, the number of stabilizers is equal for any two arrangements that appear in the same orbit, so the sum of the number of stabilizers for all arrangements in an orbit is the number of stabilizers for any particular arrangement in that orbit multiplied by the number of arrangements in that orbit, which by the Orbit-Stabilizer Theorem, is simply equal to |G|: $(\sum_{j \in \text{orb}(i)} |\text{stab}(j)|) = |\text{orb}(i)| \times |\text{stab}(i)| = |G|$.

[Relationship between number of orbits and total number of stabilizers] Thus, the total number of stabilizers in each orbit (the sum of the number of stabilizers for each arrangement in that orbit) = |G|. Suppose there are k orbits in S: orb(1), orb(2) , … , orb(k). The total number of stabilizers across all elements in S = $n = (\sum_{i \in S} |\text{stab}(i)|) = |\text{orb}(1)| \times |\text{stab}(1)| + |\text{orb}(2)| \times |\text{stab}(2)| + \ldots + |\text{orb}(k)| \times |\text{stab}(k)| = |G| \times k$. Thus, $k = n / |G| = (1 / |G|) \times n$. We have previously shown that $(\sum_{i \in S} |\text{stab}(i)|) = n = (\sum_{R \in G} |\text{fix}(R)|)$, so the number of orbits of S is

$$k = (1 / |G|) \times (\sum_{R \in G} |\text{fix}(R)|).$$

■

Conclusion

Burnside’s Lemma is an invaluable tool for counting unique arrangements because it allows us to count fixed elements (which feature predictable structure that has been organized by group theory) instead of manually listing elements and arranging them into orbits. Burnside's Lemma can be used as the basis for more complex counting machinery.

In this paper, we introduced the ideas of counting and equivalence. We described infinite frieze patterns and showed how they can be counted using Burnside's Lemma, and gave a general formula for counting a special class of frieze patterns. Finally, we gave a formal proof for Burnside's Lemma.

References

[Pisanski, et al] Pisanski, T., Schattschneider, D., Servatius, B., “Applying Burnside’s Lemma to a One-Dimensional Escher Problem”, *Mathematics Magazine*, 79(3), 2006, 167-180.

[Gallian] Gallian, Joseph A. *Contemporary Abstract Algebra*. Brooks/Cole, 2013.

Figure 1: 16 possible Arrangements:

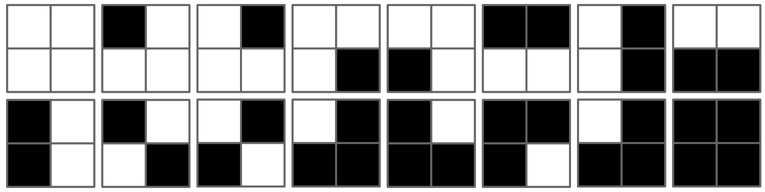


Figure 2:

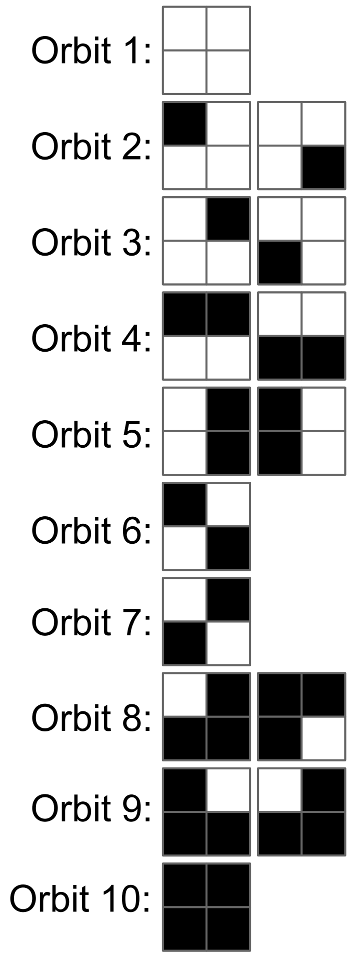


Figure 3:

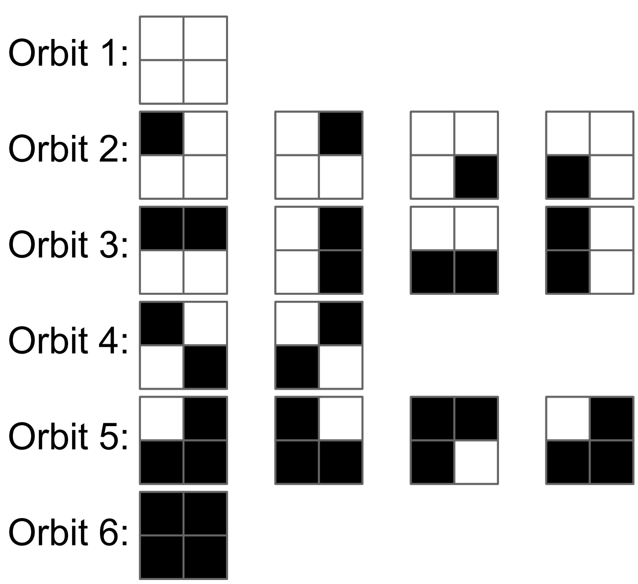


Figure 4:

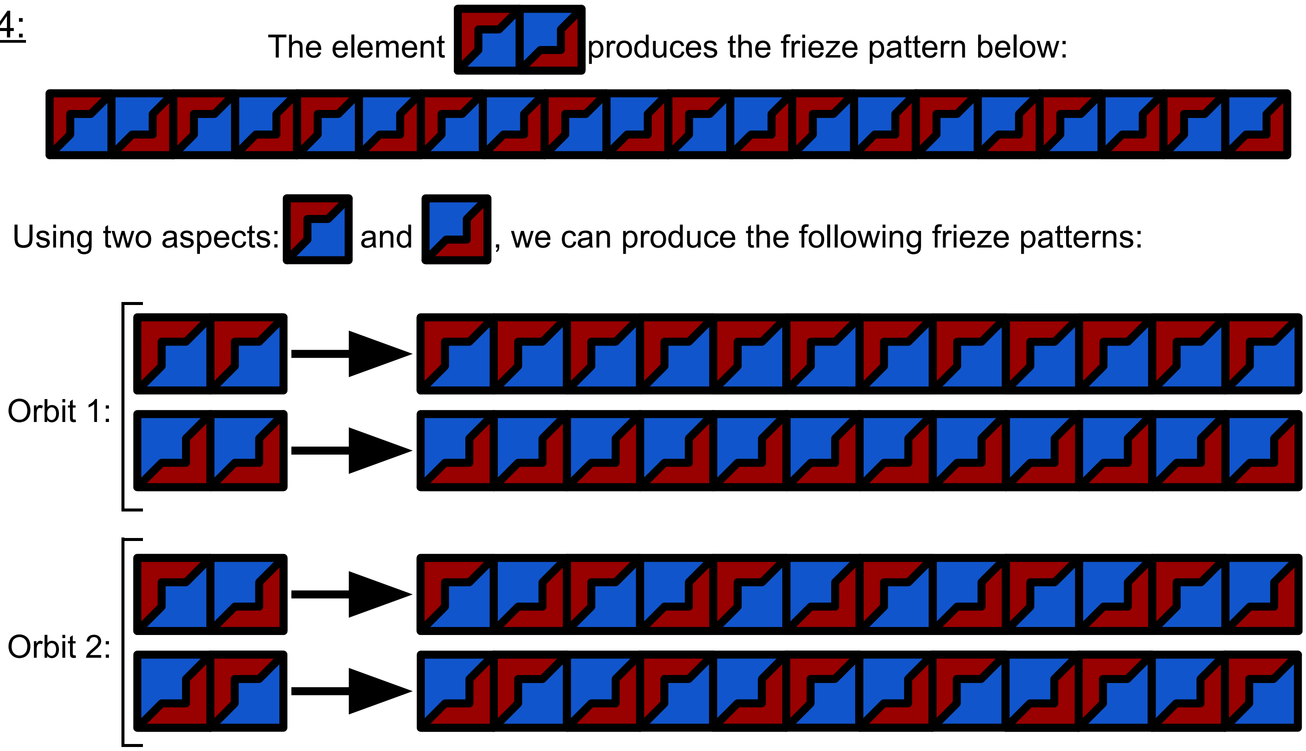


Figure 5:

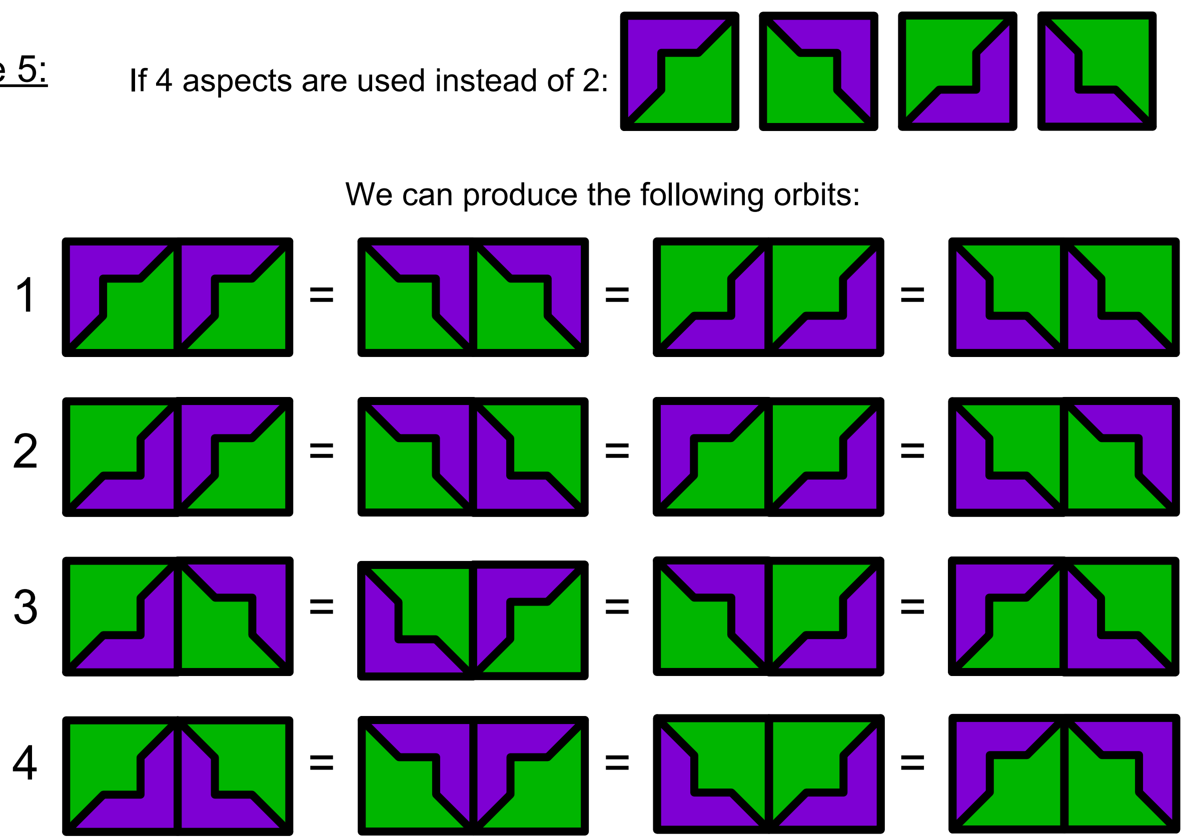


Figure 6:
All possible symmetries of a 1×6 frieze pattern

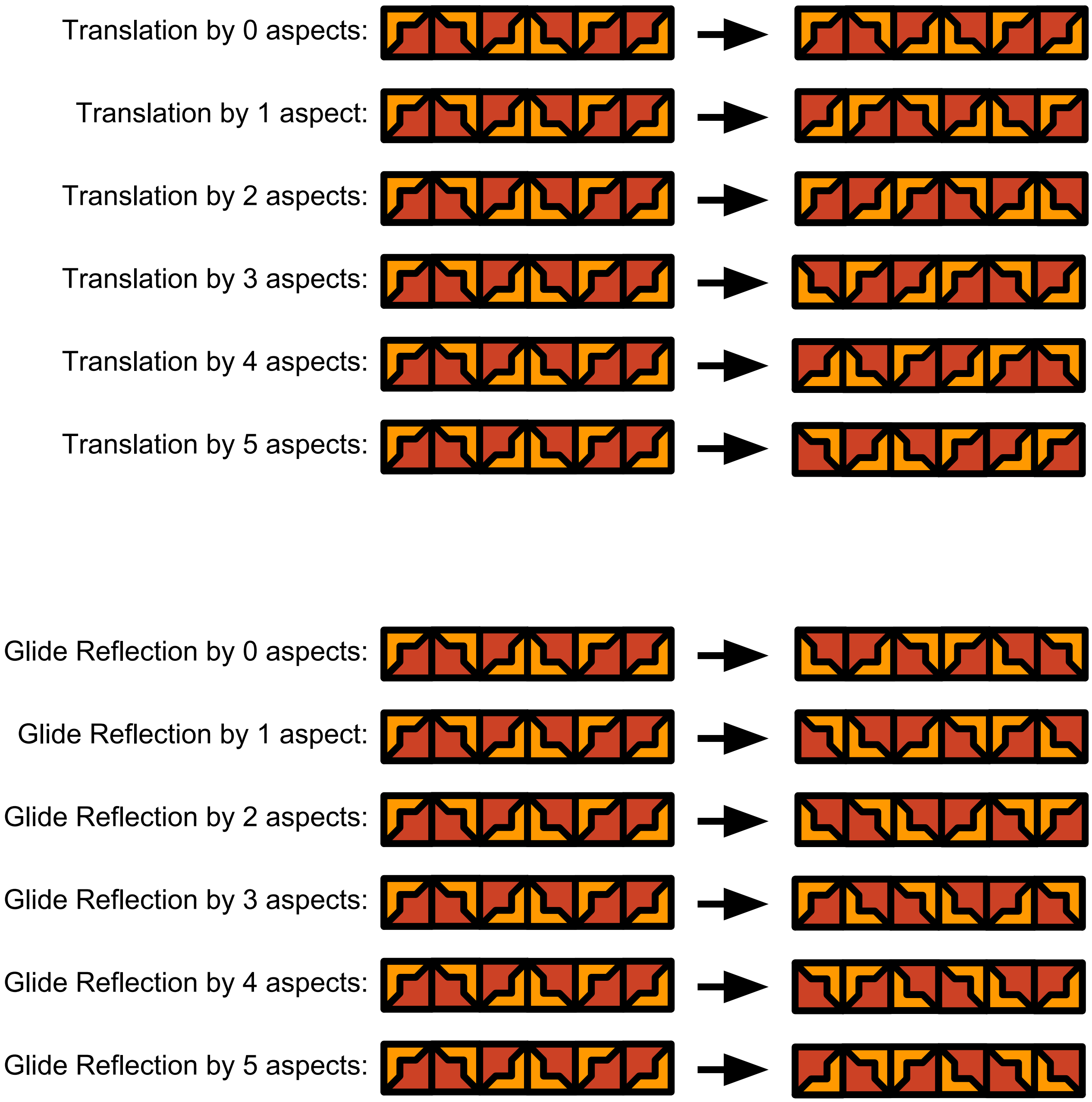
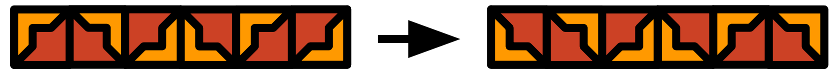
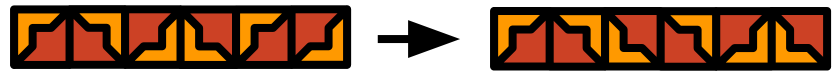
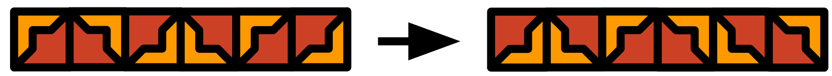
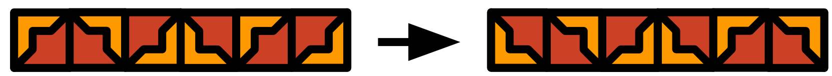


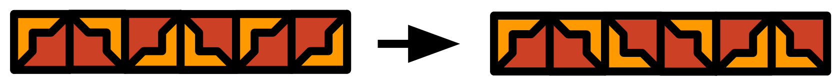
Figure 6 (cont):
All possible symmetries of a 1×6 frieze pattern

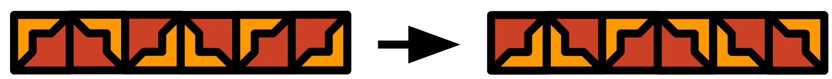
Vertical Reflection through the left border of aspect 1:


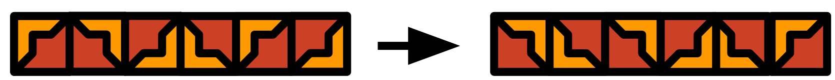
Vertical Reflection through the left border of aspect 2:


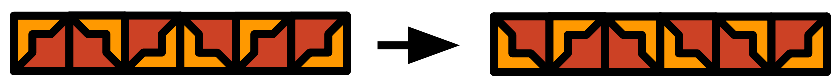
Vertical Reflection through the left border of aspect 3:


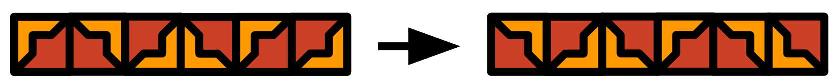
Vertical Reflection through the left border of aspect 4:


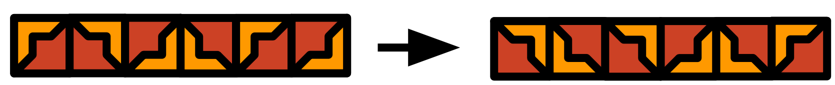
Vertical Reflection through the left border of aspect 5:


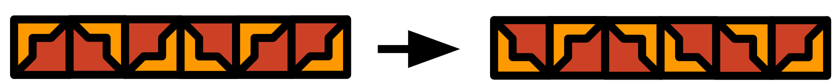
Vertical Reflection through the left border of aspect 6:


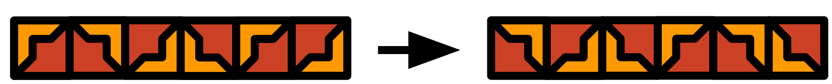
Vertical Reflection through the center of aspect 1:


Vertical Reflection through the center of aspect 2:


Vertical Reflection through the center of aspect 3:


Vertical Reflection through the center of aspect 4:


Vertical Reflection through the center of aspect 5:


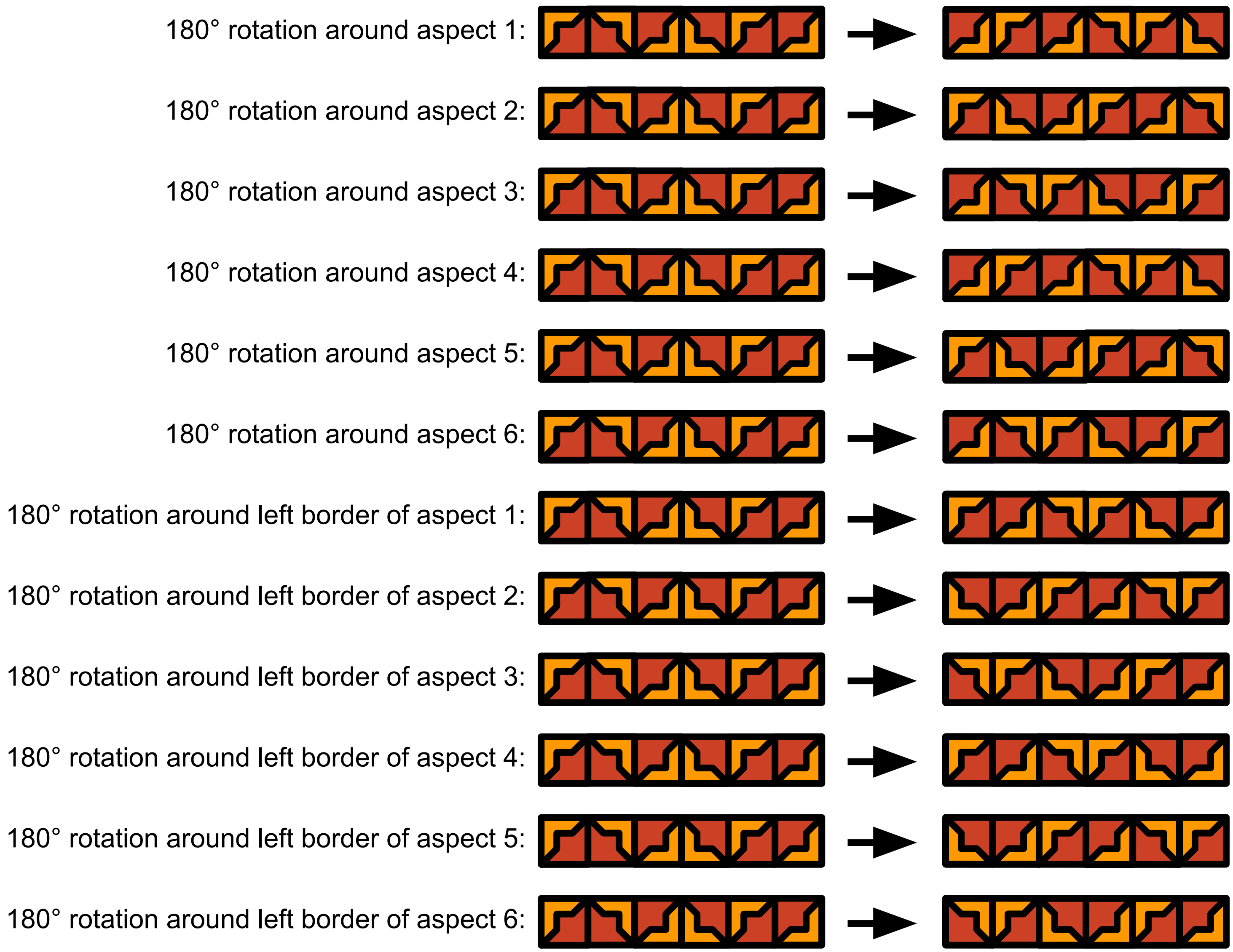
Vertical Reflection through the center of aspect 6:


LB (left border) 1 is equivalent to LB4. LB2 = LB5. LB3 = LB6.

C (center) 1 = C4. C2 = C5. C3 = C6.

Therefore, there are only 6 possible vertical reflections.

Figure 6 (cont):
All possible symmetries of a 1×6 frieze pattern



Rotation around aspect 1 is equivalent to rotation around aspect 4. Similarly for aspects 2 and 5, as well as 4 and 6.

Rotation around the left border of aspect 1 is equivalent to rotation around the left border of aspect 4. Similarly for aspects 2 and 5, as well as 4 and 6.

Therefore, there are only 6 possible 180° rotations.

Figure 6b:

This element is fixed upon reflection across the dotted line when A1 = mirror(A6), A2 = mirror(A5), and A3 = mirror(A4).

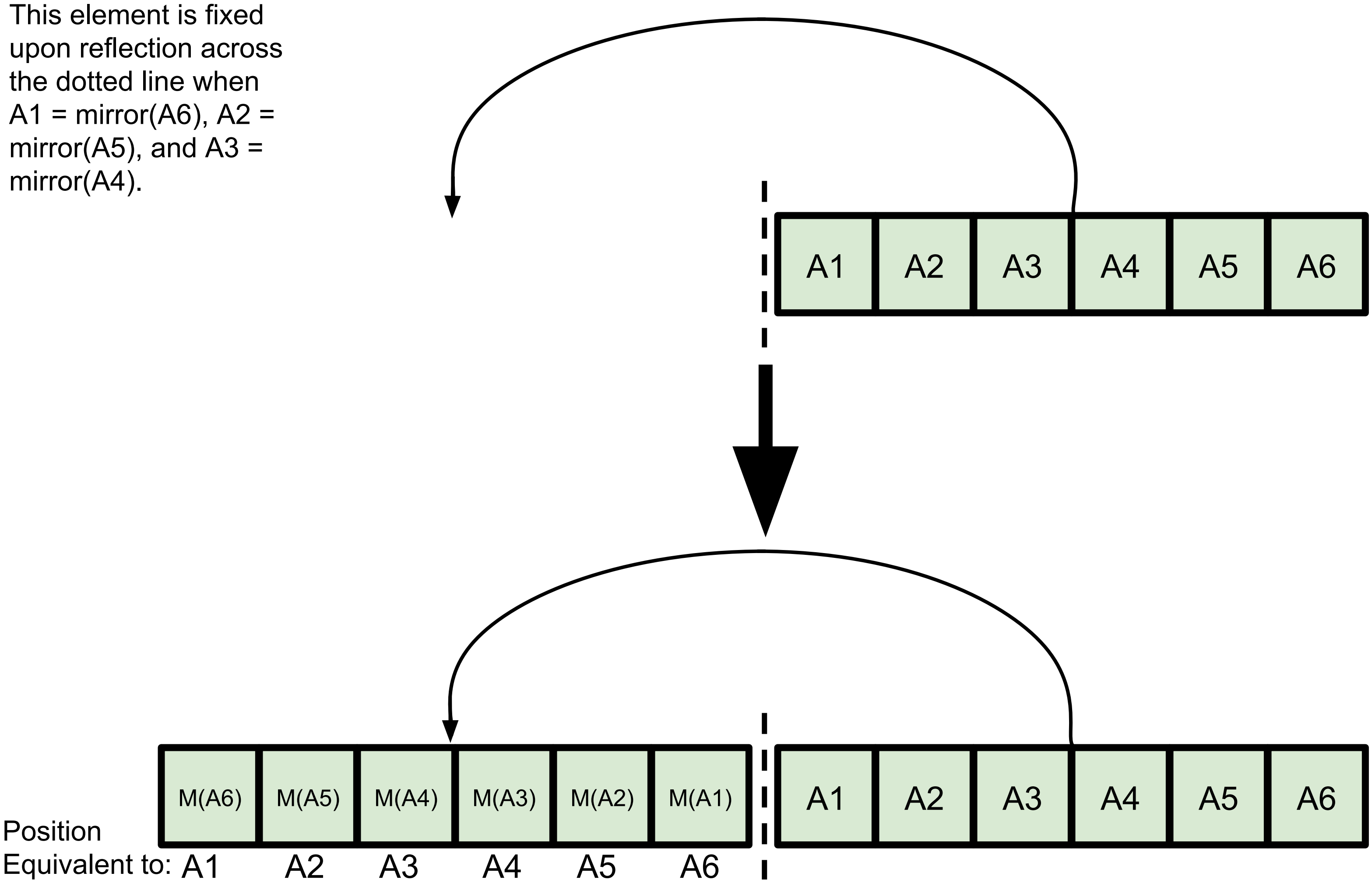
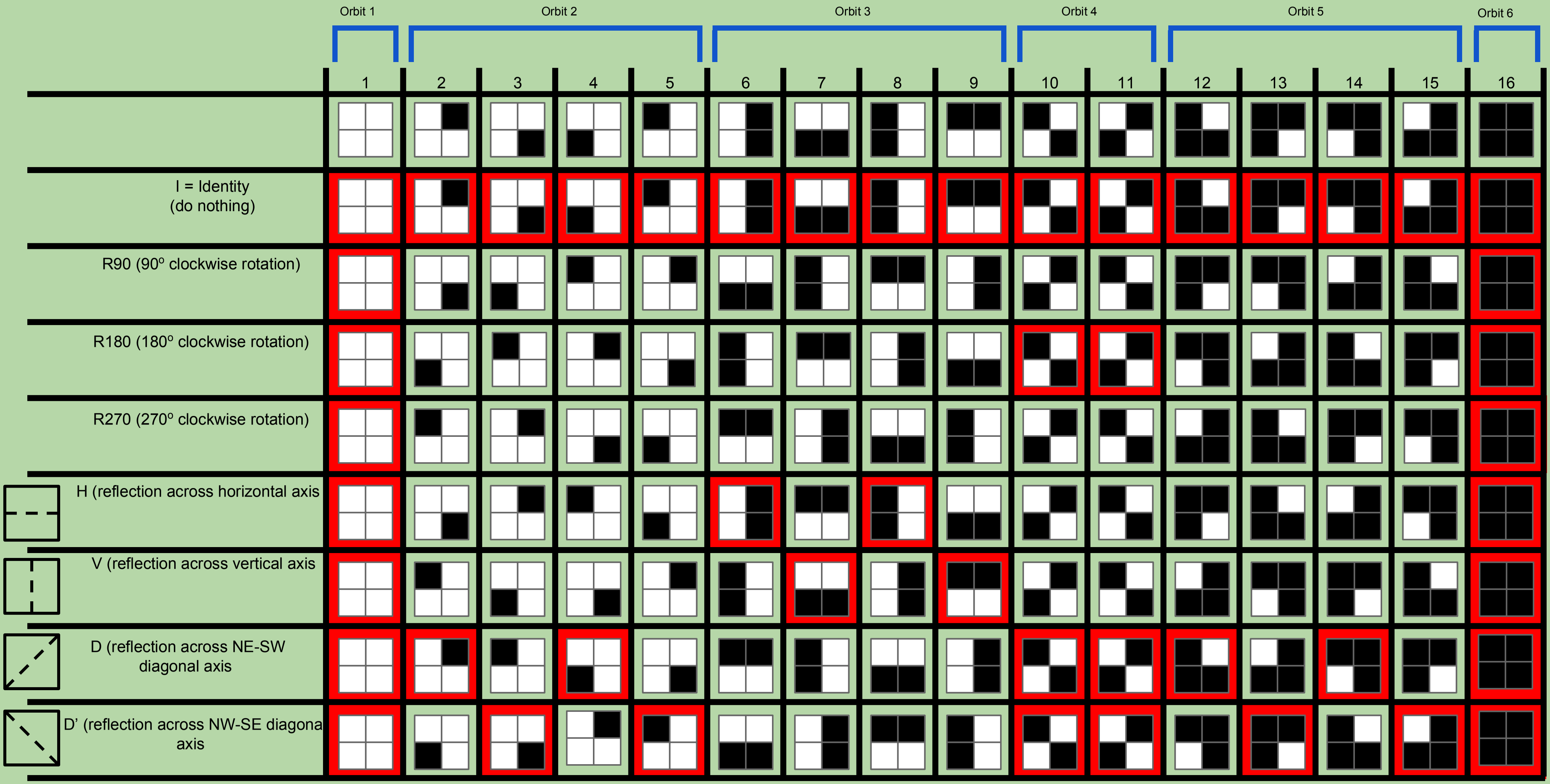


Figure 7:

Burnside's Lemma



Highlighted red cells are those where the arrangement (column) is unaffected (fixed) by the symmetry (row). Burnside's Lemma states that by summing the number of fixes for each symmetry (row), and dividing by the number of symmetries, we can find the number of orbits. In this case, (16+2+4+2+4+4+8+8) / 8 = 48 / 8 = 6 orbits. The total fixes can also be counted by arrangement (column).