Patterns in Rubik's Cube

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No cubies were harmed in the making of this honors thesis.

Abstract

The Rubik's Cube is a popular three-dimensional puzzle composed of 6 large faces, each divided into 9 colored facets glued onto 26 smaller cubes, also known as "cubies." Initially the facets on each cube face all have the same color (6 colors total), but the cubies are then permuted. The goal is to rotate faces of the (large) cube until the cubies are in the correct position along with the correct color orientation. There is an upper bound of 43 quintillion possible arrangements or permutations.

In this paper, I explore the basic algorithms used to solve the cube and the patterns within these algorithms. Most algorithms turn out to be composed of combinations of commutators and conjugates. Commutators are moves with the pattern (A B A' B'), where A and B are sequences of one or more rotations of the cube, and A' and B' are the respective inverses of those rotations. Conjugates are moves with the pattern (A B A'), where A and B are sequences of different rotations and A' is the inverse of the rotations of A. I will focus on explaining why these moves are useful, by mapping the movement, position, and orientation, of the cubies. For example, I will show that commutators made of adjacent rotations have order 6 (have no effect when repeated 6 times). I will discuss the method of concluding the total number of permutations possible in Rubik's Cube and touch upon the smallest number of moves to complete the cube, also known as God's Number. I also report on additional observations for future research.

Contents

Introduction to the Cube

[The following information on Rubik's Cube was found from the official Rubik's Cube website: rubiks.com]

The credit for the creation of Rubik's Cube is given to a Hungarian professor of architecture named Erno Rubik. In 1974, Rubik created a figure that was once deemed to be impossible. He created a solid cube that turned in multiple directions and stayed together. Rubik's original cube was constructed out of wood and he painted the sides of the cube to make it more visually appealing. Wanting to continually better himself as an educator, Rubik would search for new and exciting ways to present information to his students. He built his cube originally to show his students a representation of spatial relationships. During a lecture, Rubik mindlessly turned the layers in the cube as he was speaking to his students. When he returned home, he attempted to return the cube to the solved state but found that he could not. After a month of failed attempts, Rubik finally solved his creation and developed the beginning stages of the algorithm.

"Erno has always thought of the Cube primarily as an object of art, a mobile sculpture symbolizing stark contrasts of the human condition: bewildering problems and triumphant intelligence; simplicity and complexity; stability and dynamism; order and chaos."

Although the invention of this tool was groundbreaking, it required a combination of different strategies to get it to be what it is known as today. The first cubes were manufactured by a Hungarian company Politechnika as "Magic Cubes." These cubes were twice the weight and about 1.5x larger than the size they are today. At the time of the birth of the cube, Hungary was part of the Communist regime behind the Iron Curtain where imports and exports were extremely limited to the bare essentials.

If Rubik could not get the toy exported as a toy, he took the approach of spreading his creation as an educational tool. Mathematicians from his university took the cube to international research conferences and eventually it made it's way into toy conventions all over Europe, most importantly the Nuremberg Toy Fair in 1979. At this fair Tom Kremer, a toy specialist, agreed to make this cube one of the highest selling toys of all time. Ideal Toy Company agreed to that the project on the one condition that the name be changed. The publishing department believed that the name "Magic Cube" resembled a form of witchcraft and the toy's name was then changed after its creator to Rubik's Cube.

Since official international depute in 1980 there has been an estimated 350 million cubes sold. The cube has created a new sport called Speedcubing that includes various types of cube competitions including one-handed, blindfolded, and even with feet only. The fastest time in an official competition to solve a 3 x3 x 3 Rubik's Cube was 4.9 seconds by Lucas Etter in the fall of 2015.

There are other sizes of Rubik's Cube that have come about since the launch:

- 2 x 2 x 2 cube known as The Pocket
- 4 x 4 x 4 cube known as The Revenge
- 5 x 5 x 5 cube known as The Professor

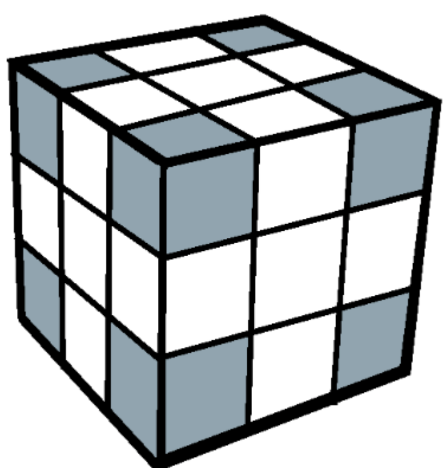
The largest functioning Rubik's Cube is 17 x 17 x17 cube whose record solving time is approximately seven and a half hours.

“The beauty of the Rubik’s Cube is that when you look at a scrambled one, you know exactly what you need to do without instruction. Yet without instruction it is almost impossible to solve, making it one of the most infuriating and engaging inventions ever conceived.”

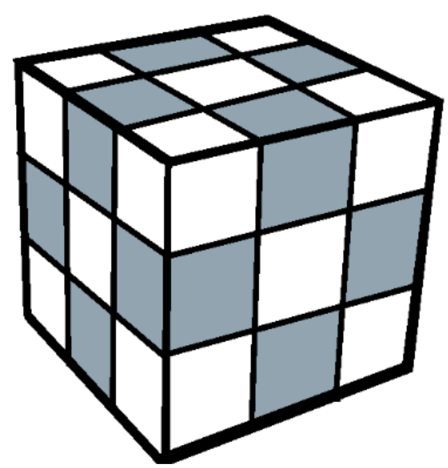
The question at hand now is how does a person get caught up in the mysteries of such a frustrating toy? When I was a freshman in high school I twisted my kneecap in gym class and ended up on bed rest for two weeks. Now this was before the Netflix and the massive social media movement so you could only imagine how slow time was passing by day after day as I sat on my couch home from school decomposing. My mother comes home from work one day and throws a brand new Rubik’s Cube at me and tells to do something productive. The toy always interested me but I had never really played around with it before. I twisted and turned the cube for a few days and got nowhere so I resorted to the Internet. After watching and following a few tutorial videos, within two days I solved the seemingly impossible toy. From that moment, from that feeling, I was hooked. When I first started my record time was approximately 20 minutes and by the time my knee was fully recovered and I was back in school, my time was just under two minutes. It became addicting immediately. It turned into a competition with myself to get my time better and better. Now, 8 years later, I could not pick up a cube for months but once I do, it’s like riding a bike. My hands just do the rotations using muscle memory. I often will play with the cube as a stress relief because I feel like solving the cube has become very therapeutic. I enjoy showing people that I can do the cube and watch the awe in their faces as I tell them that I’m part of the 5.8% of the world’s population that can solve the Rubik’s Cube.

Description of the Cube

The Rubik’s Cube is a three dimensional puzzle with an upper bound of 43 quintillion permutation possibilities. The cube is composed of 54 facets on 26 smaller cubes, also known as “cubies.” The cubies are found in 6 different colors in sets of 9 facets: green, white, red, yellow, orange, and blue. The Cube is on a fixed axis: the green center cubie is located across from blue, red is across from orange, and white is across from yellow. The center cubie of each side of the Rubik’s Cube indicates the color that needs to be solved for on that particular side. The types of cubies found on the cube are 8 corner cubes with 3 facets and 12 edge cubies with 2 facets and the center cubie with one facet. The center cubies never change their position, however they do rotate in place.



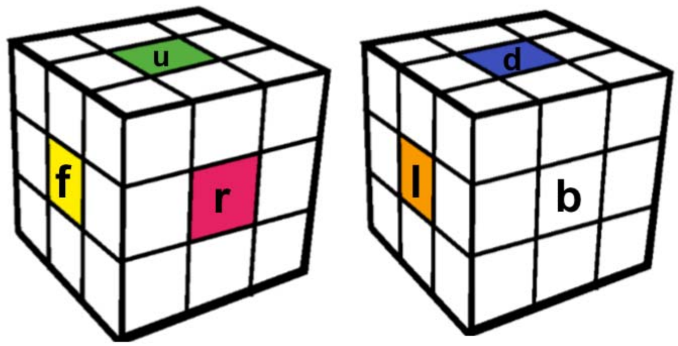
(Corners)



(Edges)

A method to labeling the face of the cube is to hold the cube using both hands. If you do not have a spare cube, refer to the diagram.

If you are holding the cube, let the green face be the one facing the ceiling and let the red face be in your right hand as the yellow face is facing your body. If you are holding the cube correctly, the orange face should be in your left hand, the white face should be facing away from you and the blue face should be on the bottom of the cube.

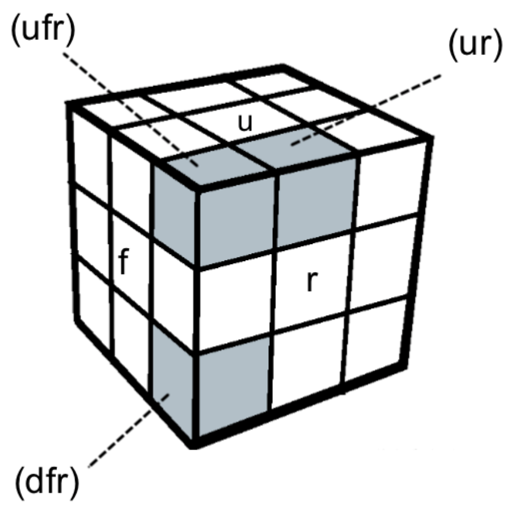


In terms of how the cube is being held at this moment:

Green face → Up (u)
Red face→ Right (r)
Yellow face → Front (f)
Orange face → Left (l)
White face → Back (b)
Blue face → Down (d)

If you move the cube around in your hands, the colors are no longer references to what side of the cube you are dealing with. The names of the sides of the cube are in reference to how it's being held e.g. whatever face is in your right hand will always be Right unless you rotate the cube in your hands in which case the new face in your right hand is now considered Right.

When referring to a particular cubie on the cube, we indicate this by a sequence of one, two or three lowercase letters. A corner cubie is notated by three letters (up/down, front/back, left/right), a two-letter sequence notates an edge cubie and a center cubie is referred to as a single letter.

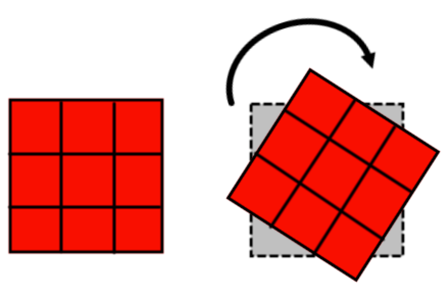


Overall there are 20 moveable pieces, however, when moved, corners will only be translated to another corner position and edges will only be translated to another edge position. Because of this property, a completed turn is a complete 90-degree rotation. Any rotation less than 90 degrees, not including 0 degrees, is considered incomplete and you cannot continue to solve the cube. Each rotation of a center layer is equivalent to rotating the two adjacent face layers. A solved cube has all the cubies in the correct color position, matching the center cubie color, along with the correct orientation.

The movements in a cube are noted by the face orientation of the cube.

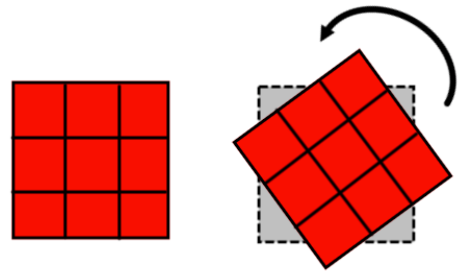
The rotations are noted by:

- U (up)
- D (down)
- F (front)
- B (back)
- L (left)
- R (right)



This notation indicates a 90-degree clockwise rotation.

The inverses of these rotations are noted by:



- U' (up inverted)
- D' (down inverted)
- F' (front inverted)
- B' (back inverted)
- L' (left inverted)
- R' (right inverted)

This notation indicates a 90-degree counterclockwise rotation.

Summary:

1. A single lowercase letter refers to a specific face;
2. A sequence of lowercase letters refers to a particular cubie position;
3. A single uppercase letter refers to a rotation of a particular face either clockwise or counterclockwise.
4. A sequence of uppercase letters refers to a series of rotations of those face layers.

The Complete Basic Beginner Algorithm

Rubik's Cube is solved with the application of an algorithm composed of a set of move sequences given particular pattern occurrences. Many speedcubers have taken the basic move sequences and modified them to allow them to do multiple steps simultaneously. The basic algorithm that beginner cubers use involves moving particular cubies in place before being able to move other cubies because of the effects the move sequences have on the cube has a whole.

The following steps are done in the order below. The move sequences focus on the movement of particular cubies and some of them change the position and orientation of other cubies on the cube. At certain times, those additional movements are not of concern because they are affecting cubies that aren't of importance at that moment.

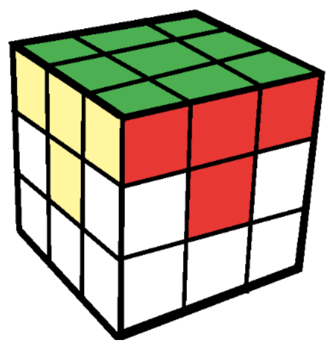
[The following algorithm was based from the booklet that is given with the purchase of an official Rubik's Cube and can also be found on youcandothecube.com]

Step 1: Top Cross

Although you can start with any color, I start with the green face. The first step is to get all the edge cubies with a green facet to the green face. This is done by examining the cube and locating the edge pieces with a green facet. This step doesn't involve any specific move sequences. The purpose of this step is to use intuition to move the layers to obtain the following image. Because it is the first step of solving the cube, we aren't concerned on the movement of other cubies.

Step 2: Top Corners

After the green edge faces are placed into position we have to place the corner cubies with green facets into place. Because the corner cubies have three facets, you have to match the corner cubie with the colors of the adjacent centers. You place the cubie in the (**dfr**) position and apply R' D R to bring the cubie to the (**ufr**) position. You repeat the move sequence R' D R D' as needed to orient the cubie correctly. This sequence does not affect the position of the edge cubies in the green layer.



*Note: After this is completed, the cube is rotated 180 degrees where the blue face is now located in the Up position. I will continue to refer to the blue face as the bottom layer however the following algorithms are referring to the blue face as Up and the green face as Down.

Step 3: Middle Layer

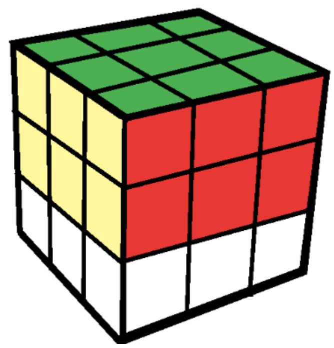
Now that the green layer, the top layer more specifically, is completed, the next step is to place the edge cubies in the correct placement based on the colors of the cubies. This step again involves examining the cube to locate the desired cubie. You place the cubie in the (**uf**) position and apply one of the following:

Move to (**lf**): U' L' U L U F U' F'

Move to (**rf**): U R U’ R’ U’ F’ U F

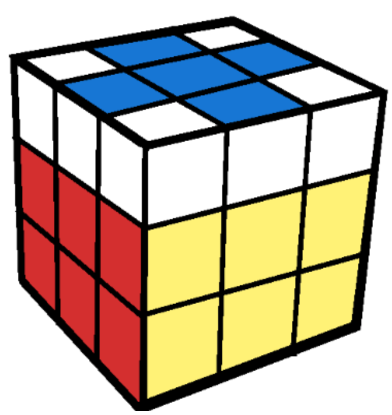
You apply either one of the sequences once per edge cubie. Repeat this process for all the necessary edges in the middle layer.

These move sequences do not affect any of the cubies in the layer with the green facets or any already correctly placed edge cubies in the middle layer.



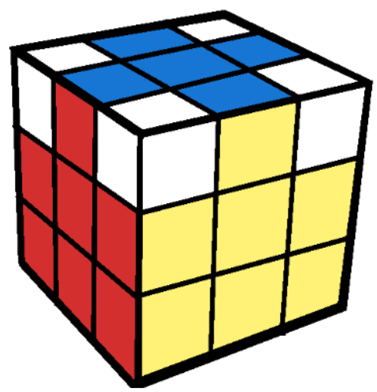
Step 4: Bottom Cross

Now that the top and middle layers are finished all the cubies with blue facets are in the bottom layer. To solve the layer we must first get the blue cross by orienting the blue edge cubies to proper position by applying F R U R’ U’ F’. If there is one blue facet facing the Up layer, apply the sequence 3 times. If you have two edge blue facets in the shape of an L on the layer, apply the sequence twice. If you have two blue edge facets in a straight line, apply the sequence once. If the cross is already present, you can skip this step. This move sequence is different that the green cross because now we are mindful of the other cubies in the cube. This move sequence does not affect the top or middle layer.



Step 5: Proper Edge Position

After the blue edge cubies have be placed all facing the blue face, we must shuffle the edge cubies to proper color position with the adjacent sides with R U R’ U R U U R’. After this the blue edge facets will match the adjacent colors of the other layers. This sequence should be repeated until necessary. This sequence does not move anything other than the 4 edge cubies on the bottom layer.

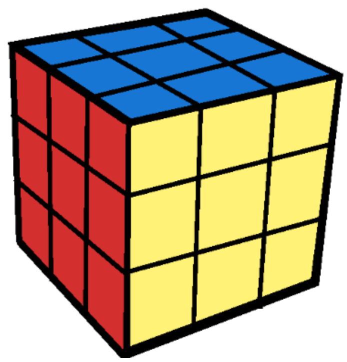


Step 6: Corner Piece Position

At this point in time, the only pieces incorrectly placed are the four corner cubies. To rotate the corner cubies to map to each other’s positions, we use U R U’ L’ U R’ U’ L and repeat as needed until the cubies are in the correct position, but do not necessarily have the correct orientation. This affects only the following three corners: (ubl ubr ufl). The (ufr) corner is not affected. It is recommended that if a corner cubie is already in the correct position, you should hold the cube with that cubie being in the (ufr) position.

Step 7: Four Corner Orientation

To orient the corner cubies in the bottom layer we apply R’ D R D’ repeatedly until the (ufr) cubie is correctly oriented. The next step is to perform U and repeated the move sequence to orient the cubie that is now in the (ufr) position. You repeat this process until the cube is solved. This sequence will look like it messes up the entire cube initially but it actually does not affect any other cubies but these four.



The Permutations of the Cube

When the Rubik's Cube was first introduced to the world as a toy, the advertisements claimed that there were over 3 billion possible arrangements of the cube [Bandelow]. This number was established by a number of different researchers who were trying to manually come up with every possible permutation. In reality there is an upper bound of 43 quintillion possible permutations [Ferenc]. To put this in perspective of how large the number is, if you turned Rubik's Cube once every second it would take you 1400 trillion years to go through all the configurations. This number is based on the limitations and characteristics of the cube itself.

[Information in the following section was based on the work of Scoot Vaughen, Miami Dade College North Campus.]

At first glance, one would jump to the conclusion that the Rubik's Cube is formed by 27 cubies, however there are only 26 cubies because there is no cubie in the center of the cube. All eight corner cubies have 3 facets each, all twelve edge edges have 2 facets each and the 6 center cubies have 1 facet each and are also fixed on the axis, as mentioned before.

Because the cube has 6 faces and each face has 9 facets that gives us a total of 54 facets. A first upper bound on the number of permutations for the cube is 54!, roughly 2.3 x 10^71. This approximation is made using the fact that there are 54 different facets on the cube. The approximation assumes that each facet moves individually without affecting the others. Basically, a permutation total of 54! would arise if you were allowed to peel the stickers off the cube and place them randomly on the cubies, which is considered an illegal move. There are limits and restrictions on the moves possible on the cube, which lowers this permutation total [Rick].

Let's continue the notion that we can peel the stickers off the cube, meaning that we aren't focused on the limits of possible moves or orientations of the cubies. The approximation of 54! is too high but with the knowledge of the number of facet colors and quantity of each, we can narrow the total number of permutations. The cube is composed of 9 identical stickers for each of the 6 colors found on the cube. Using the green facets as an example. In the 54! upper bound, we are considering each of the 9 green facets as different configurations but because they're the same color, they are indistinguishable. We can divide by 9! (the number of ways the 9 facets can be interchanged) and repeat this process for each of the six colors to focus only on distinguishable configurations. Our new total permutations is narrowed to $(54!)/(9!^6)$.

This number is still too large, because we cannot peel the stickers off of the cubies. Most of the permutations in those approximations are impossible using legal moves. We want to the count of total LEGAL permutations of the cube using LEGAL moves.

As mentioned earlier in the paper, corner cubies will only map to corner cubies, edge cubies will only map to edge cubies and center cubies will only map to themselves (because they are on a fixed axis). Disregarding orientation, there are 8! permutations of corner cubies and 12! permutations of edge cubies. Center cubies, as we mentioned earlier, do not move.

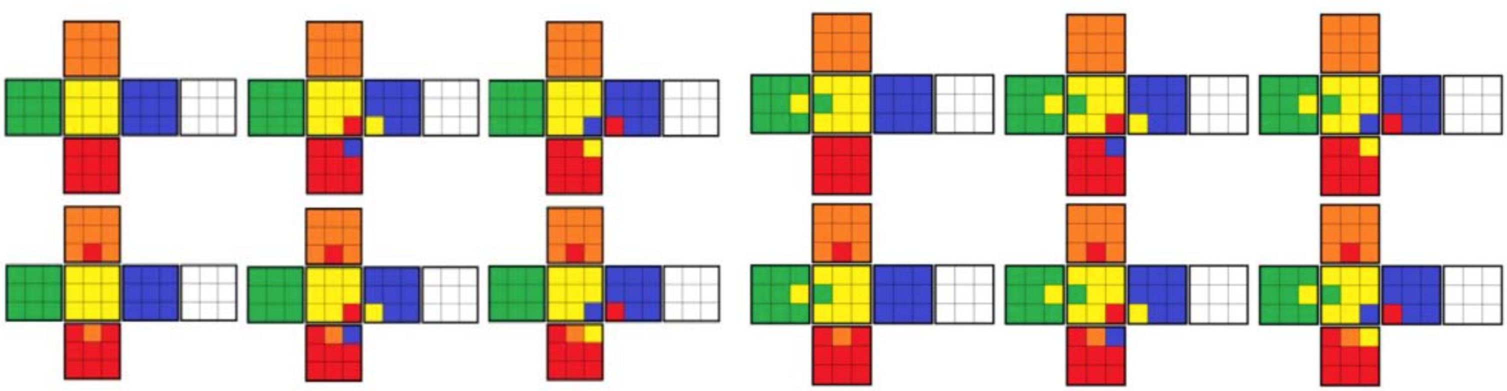
There are eight individual corner cubies with distinctive stickers that make them each different from another. One can arrange the corner cubies 8! ways, as mentioned. Because each corner has three facets, each of the eight corner cubies can be oriented 3 different ways, which gives us 3^8 orientations altogether. The total number of combinations of corner cubies is 8! x 3^8.

This process can be repeated for edge cubies:
There are twelve individual edge cubies with distinctive stickers that make them each different from another. One can arrange the edge cubies 12! ways, as mentioned. Because each edge cubie has two facets, each of the twelve cubies can be oriented 2 different ways, which gives us 2^{12} orientations altogether. The total number of combinations that edge cubies can be found on the cube is 12! x 2^{12}.

The upper bound on the total number of permutations of the Rubik's Cube can now be stated to be 8! x 3^8 x 12! x 2^{12}.
Note: The following section is a topic for further research in the future and represents my current understanding of how imagining dismantling the cube affects the possible permutations:
Up until now, we only discussed the possibility of removing the stickers and rearranging them back on the cube as an illegal move. Dismantling the cube, meaning taking the

pieces off the axis is also considered an illegal move. If you dismantle the cube, there are 12 possible sets of ways to build the cube. In other words, If you disassemble and reassemble the Rubik's Cube at random, the probability that the Cube can be solved is 1/12. Rubik's Cube's arrangements can be categorized as a series of different face turns. There are particular limitations to the movements of the cube that make certain states impossible by legal moves.

There are twelve states that the cube can be found in after being disassembled and reassembled [Maingi]. These are shown below in nets of the cube.



[Image: Mathstackexchange]

The nets are actually not color specific and can be redrawn using other colors, consistent with the diagrams above.

It is not possible to perform a legal move or move sequence that only affects a single cubie on the cube [Vaughen]. The nets above depict rearranging the cubies themselves by physical force. This is an illegal move but it must be taken into consideration. These states create entirely new set ups of the cube and in only one of those states can the cube be solved by a series of legal moves (the first net). If the cube is initially set up as one of the other 11 states, the cube can no longer be returned to a completely solved state without another illegal move. The previous upper bound involved all the possible permutations in all 12 nets and since only 1/12 of the states are viable, we divide the approximation by 12 to get:

$(8! \times 3^8 \times 12! \times 2^{12}) / 12 = 43 \times 10^{18}$

= 43 quintillion total possible permutations.

There is another way to find a weaker upper bound.

We consider the permutation group of the cubies, ignoring orientation. As described above, the total number of permutations is 8!12!. We show that all the valid permutations must be even, which will show that at most half of these permutations are valid.

[The following explanation is taken from [Gallian]]:

A permutation that can be expressed as a product of an even number of 2-cycles is called an even permutation. A permutation that can be expressed as a product of an odd number of 2-cycles is called an odd permutation.

The set of even permutations in S_n forms a subgroup of S_n, where

S_n is the group of all permutations of n elements. The group of even permutations of n symbols is denoted by A_n and is called the alternating group of degree n.

<u>Theorem</u> For n >1, A_n has order n!/2.

Proof. For each odd permutation a, the permutation (12)a is even and, by the cancellation property in groups, if (12)a = (12)b then a = b. Thus, there are at least as many even permutations as there are odd ones. On the other hand, for each even permutation a, the permutation (12)a is odd and if (12)a = (12)b then a = b. Thus, there are at least as many odd permutations as there are even ones. It follows that there are equal numbers of even and odd permutations. Since $|S_n| = n!$, we have $|A_n| = n!/2$.

Using a similar proof, one can show the following general theorem.

<u>Theorem</u> Any group of permutations which are not all even consists of half even permutations and half odd permutations. □

We now apply these facts to the cubie permutation group.

<u>Theorem</u> Every cubie permutation is even.

Proof. We first show that the rotation of any face gives an even permutation. We can use any face, so we focus on the (u) face undergoing the move U'. Using the cubie labels we can write this as a product of disjoint cycles. The picture shows the result of U'.

ubl	ub	ubr
ul	u	ur
ufl	uf	ufr

→

ubr	ur	ufr
ub	u	uf
ubl	ul	ufl

Writing the permutations in cycle notation we get the product of two 4-cycles, one for the corners, one for the edges: (ubr ubl ufl ufr) (ub ul uf ur).

If we write this as a product of 2-cycles we get:

(ubr ubl) (ubl ufl) (ufl ufr) (ub ul) (ul uf) (uf ur),

which has six 2-cycles, and so is even.

As shown, the corners and edges cycle separately because of the characteristics of the cube that was discussed previously. The cycles are odd individually, however, put together, become even. Thus, a single layer (face) rotation is an even permutation. Since every move sequence is a product of face rotations, every cubie permutation is a product of even permutations and so is even. □

The number of potential cubie is 8! X 12!, but only half of the permutations are possible. Including the consideration for the orientation we get an upper bound of

$(8! \times 3^8 \times 12! \times 2^{12})/2$.

We have not yet found a reference for a lower bound on the possible permutations of the Rubik's Cube but hope to do so at a future time.

Conjugates and Commutators on the Cube

When you think about the fact that there are only about 7 move sequences to solve this mysterious toy it's almost humbling. Of course this algorithm wasn't easy to find, but looking at it now and examining its properties, such as the way that the cubies behave under the movements, the algorithm makes more sense. The conjugates and commutators found throughout the algorithm are essential to solving the cube. A *conjugate* in a group has the form XYX'. A *commutator* in a group has the form XYX'Y'.

For example, the conjugate R' D R is used when solving the first layer of the cube, to correctly place the corner cubes in their respective places.

When solving for the top layer corners, you can first place a specific corner cubie in the (**ufr**) corner without affecting the position of any other cubies on the layer. In order to do this, you must first place the specific cubie into the (**dfr**) position and apply the conjugate.

After the initial R' movement, the cubie is in the (**dbr**) position.

After the D movement, the cubie is the (**dfr**) position.

After the R movement, the final movement, the cube will be in the (**ufr**) position.

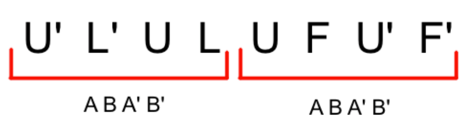
The commutator R' D R D' is first used in step 2 and step 7 in the algorithm. This conjugate will place the corner cubie into the desired position, however not necessarily in the correct orientation. In order to rotate the cubie so that it is correctly oriented, you can apply the commutator R' D R D'. Note: You use the commutator only have you finishing applying the conjugate: first R' D R then R' D R D'.

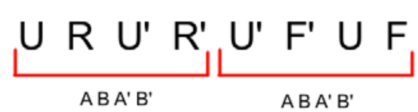
In general, commutators are found throughout the algorithm because they only affect a small number of cubies and leave everything else intact.

A commutator A B A’ B’, where A and B can be a single rotation or a sequence of rotations is often used as follows [Heise]:

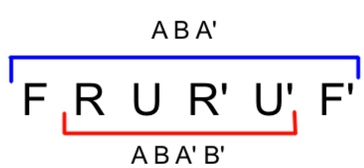
- Use A to accomplish a goal with a particular cubie.
- Use B to move another cubie into the spot where A was.
- Use A’ (inverse of A) to undo all the damage caused by A on the cube.
- Use B’ (inverse of B) to put all cubies that should not have been affected back into position.

In Step 3, we can see another occurrence of a commutator in a move sequence when solving for the middle layer. Step 3 includes two different move sequences and both of them are composed of two commutators each.



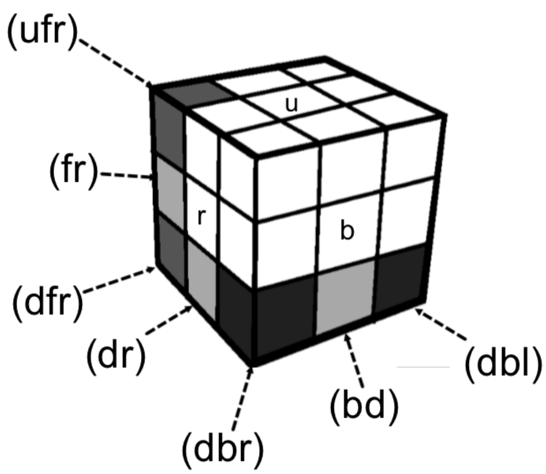


The move sequence in step 4, solving for the bottom cross, is a special case where there is a commutator within a conjugate.

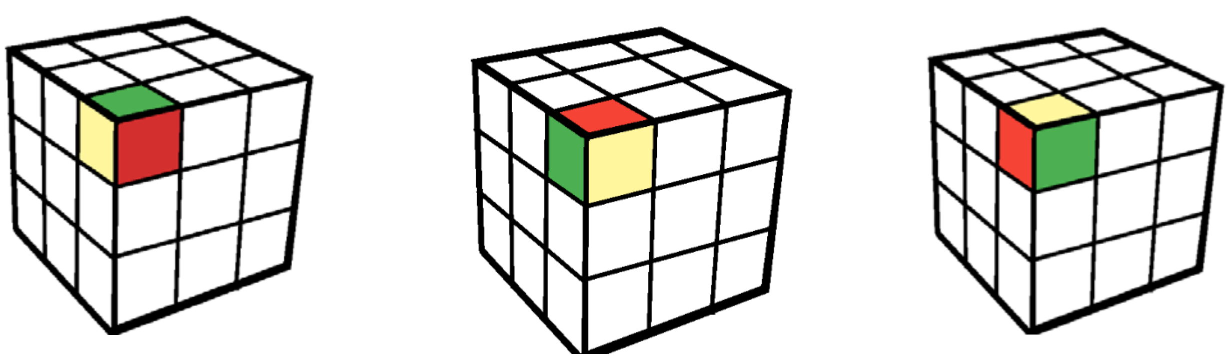


Face Rotation Commutators

The commutator R' D R D' is used repeatedly, primarily to change the orientation of the (**ufr**) corner when solving the Rubik's Cube without affecting other cubies in the up layer. In addition to rotating the (**ufr**) cubie, it also rotates the (**dfr**) cubie as well as the (**dbr**) and (**dbl**) cubies. The cubie permutation given by R' D R D' is (ufr dfr)(dbr dbl) (fr dr bd): two pairs of corner cubies are transposed, and three edge cubies are permuted.



One can observe that when applying the commutator once, the two sets of corner cubies transpose to each other's positions. After two applications, the corner cubies return to their original positions, however they have been rotated 120° counterclockwise. Below displays the sequences on the green-yellow-red (ufr) corner:



(Start) (2) (4)

Using this observation, one can conclude that after three applications, the corner cubies will transpose positions again. After four applications, the corner cubies will return yet again to their correct positions and another 120° counterclockwise rotation will be added, resulting in a 240° rotation from the original state. After six applications, the corner cubies will be in their correct position and will have had a 360° counterclockwise rotation applied. In other words, the corner cubies will be returned to their original state. The order of both the corner cubie cycles is 2. However, taking account the 120° counterclockwise rotation of each cycle, the order of the commutator will be at least 6. The edge cubie cycle (**fr dr bd**) operates at the same time as the corner cycles. With each application, the edge cubies cycle once.

After one application: (**fr**) → (**dr**)

(**dr**) → (**bd**)

(**bd**) → (**fr**)

This can be concluded also by observation.

We can also observe that after three applications of the commutator, the cycle will return the edge cubies to their original position and orientation. With this behavior, we can conclude that the edge cubie cycle has an order of 3. Applying this cycle once will act as the identity within the edges, so we can say that applying it twice will also do the same. The order of the move sequence, including the edge and corner cubie cycles, is 6.

No other cubies are affected if the commutator is performed correctly.

One can verify the effects of the commutator discussed by performing the algorithm on the Rubik's Cube or by creating a map of the cycles. As stated in the section on permutations, considering just the cubie permutations and ignoring orientation, the commutator can be written in cycle notation as: **(ufr dfr) (dbr dbl) (fr dr bd).**

One can make the observation that the corner cubie cycle has an order of 2 and the edge cubie cycle has an order of 3. The order of a *product of disjoint cycles* is equal to the *least common multiple* of the orders of the cycles that form it, i.e., the least common multiple of the lengths of the cycles [Gallian]. Since the least common multiple of 2 and 3 is 6, this also supports the conclusion that the complete commutator permutation has an order of 6 when applied to the cube [see also Chen].

We can extend this to show that any commutator X Y X' Y' where X, Y are single rotations of adjacent face layers on the cube, such as R' D R D', has an order of 6. There are three other ways to order commutators involving R then D: R D R' D', R D' R' D, R' D' R D. All three of these commutators also involve the transposition of two sets of two corner cubies and a cycle of three edge cubies. The proofs for each of these are similar to the proof above using the following cycles:

R D R' D':	(dfr dbr) (ubr dbl)	(dr bd br)
R D' R' D:	(dfr dfl) (ubr dbr)	(df br dr)
R' D' R D:	(ufr dfl) (dfr dbr)	(df fr dr)

Because of the symmetry of the cube, the same will hold true for any adjacent layers.

God's Number

[The following section is based on material from [Garron].]

A common question that has arisen from solving the cube is "what is the minimum possible number of moves sufficient to solve every Rubik's Cube?". This number is called God's Number. God's Number research originally began with Erno Rubik himself but because of all possible permutations of the cube (discussed previously), it was nearly impossible for him to figure it out manually.

In July 1981, Morwen Thistlewaite estimated a lower bound of 18 and an upper bound of 52. Twenty-nine years later, July 2010, Tomas Rokicki, Herbert Kociemba, Morley Davidson, and John Dethridge proved that God's Number is exactly 20. The question now is how they came to this conclusion? Below, I will list the steps they took to show the number is at most 20.

How did they solve all 43,252,003,274,489,856,000 positions of the Cube?

- They partitioned the positions into 2,217,093,120 sets of 19,508,428,800 positions each. [This information was taken directly from Garron however with this partition they get 4.3 x 10^{18} not 43 x 10^{18}.]
- They reduced the count of sets we needed to solve to 55,882,296 using symmetry and set covering.
- They did not find optimal solutions to each position, but instead only solutions of length 20 or less.
- They wrote a program that solved a single set in about 20 seconds.
- They used about 35 CPU years to find solutions to all of the positions in each of the 55,882,296 sets. (Rokicki)

Next Steps

With Rubik's Cube being one of the most puzzling mathematical tools, this research can be extended in multiple ways.

(1)

Much of this paper focused on the movements of the corner and edge cubies. Although it was stated that the center cubies never change positions because they are set on the axis, or the backbone of the cube, they do rotate in place. The center cubies have only one facet, so no matter how often and in what directions the cubie turns, it creates no difference when solving the cube. The only way of noticing the center cubie's rotation is by marking it with a marker. In future work I would like to understand why the center cubie turns on its axis and how that affects the rest of the cubies during movements. How can the cube be completely solved taking the orientation of the center cubie into consideration?

(2)

Since it was possible to find the order of the face-rotation commutators discussed previously, I would like to extend the research into looking into the order of different kinds of move sequences that are possible. I would focus on sequences that include either a conjugate or commutator, and to somehow relate it back to the results on face rotations.

(3)

Rubik's Cube (3x3x3) and the Professor Cube (5x5x5) are solved very similarly because they share many of the same properties. I would like to look into the effect of the commutators from the Rubik's Cube on the Professor Cube and see how they are related. Also, I would like to figure out the number of possible configurations of the Professor Cube and see how this scales with the number of configurations of the Rubik's Cube.

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